Investigation 2.1

POLYOMINOES

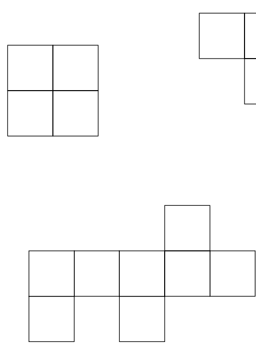
A mathematical term for features or relationships that many objects have in common is *invariant*: while the object may vary, the feature does not.

To understand science or social relationships, mathematics or the arts, history or psychology, one must look at how things differ and *also* at how they are the same. Mathematics looks at quantities, relationships in space, ways of classifying things, and processes by which things are done. Our focus is on shape: "What does it mean for two shapes to be the same?"

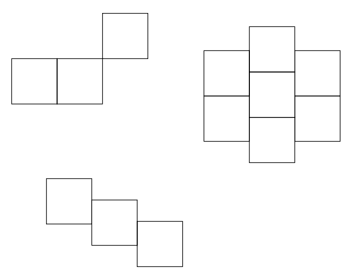
Mathematical language tries to be clear and precise, specifying exactly one meaning for each word. Sometimes this means inventing new words, or specializing the meanings of familiar words. Either way, the process involves people getting together and agreeing on what they will mean by the words they use.

Here is a chance for you and your classmates to take a familiar but ambiguous term—"the same"—and make it clear by deciding what you will (and will not) mean by it.

One way to create shapes is to start with simple elements and combine them to build more complex objects. Polyominoes are shapes that are built of squares, following one rule: when two squares are joined, their edges must fully coincide. So, the three shapes on the left are polyominoes; the three shapes on the right are not, because the squares don't meet edge-to-edge.



Polyominoes



Not Polyominoes

You might draw the figures on graph paper, or use cutout squares that can be arranged and then taped together. Most of these problems are best done by working with classmates; that will help you find more combinations.

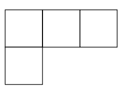
1. **Dominoes** How many different figures can be made from just two squares combined according to the rules?

2. **Triominoes** How many different figures can be made by connecting exactly *three* identical squares edge-to-edge and vertex-to-vertex? (If two figures are mirror images of each other, consider them the same.)

3. **Tetrominoes** How many different figures can you create this way using four squares?

It is easier to talk about things when they have names. The authors of this book have come up with names for a couple of the tetrominoes. Your class may want to name all of them and post pictures with the names so that it's easier to discuss solutions to the next few problems.

This shape is what we call the "L tetromino":



4. Create a shape by putting an L tetromino together with one that is not an L. Let's call any eight-square shape made in this way a "squilch." Can you make the same squilch from an L and some *other* tetromino?

5. How many different ways can you make your squilch using an L and one other tetromino?

Seriously preparing your explanation usually brings you new insights.

6. Pentominoes are made by combining *five* squares edge-to-edge in the plane. How many different pentominoes can you find?

CHECKPOINT..................................

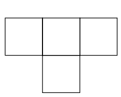
In the process of deciding whether some of these pentominoes were actually different from one another, did you find it easier to draw the figures or to use cutouts? Explain why.

7. Explain the naming convention for the polyominoes (domino, triomino, tetromino, pentomino, . . .). What word would you choose to describe polyominoes made from *ten* squares?

8. Describe how you can decide if two pentominoes are the same or different.

TAKE IT FURTHER..............................

This shape is what we call the "T tetromino":



9. Make an eight-square shape by combining the T with some other tetromino. Then choose *another* tetromino. How many ways can you make your eight-square shape by combining the T with this new tetromino?

10. Is there any tetromino that combines with another so that you can't make the same shape in two different ways (including using that tetromino with the others)?

11. It is easy to cover an 8×8 checkerboard with 16 of the "square" tetrominoes (the 2×2 shape). Is there any tetromino that won't cover a checkerboard (using only itself)?

12. It is impossible to cover a checkerboard with one square tetromino and 15 Ts, or 15 Ls, or 15 of the other two tetrominoes. You can cover a checkerboard using 2 squares and 14 of another shape. Can you find two of the other four shapes that will do this?

PENTOMINO PROBLEMS

Twelve different pentominoes with five squares each makes a total of sixty squares.

13. Find a way to arrange twelve different pentominoes into a 6×10 rectangle. Try to make a 5×12 rectangle also. Which of these problems do you think has more solutions?

14. Explain why it isn't possible to arrange the 12 pentominoes into a 2×30 rectangle.

The reason why there are only two solutions for the 3×20 and 2339 for the 6×10 can be sought in a field of mathematics called *combinatorics*.

The number 60 has more factors than the six just used in the problems above. Rectangles of dimensions 3×20, 4×15, 5×12, and 6×10 may all be made from the twelve pentomino pieces. For the first rectangle, there are only two solutions; yet there are 2339 different ways to make a 6×10 rectangle.

15. Using four different pentominoes, create two identical ten-square shapes.

Challenge: Can you explain why four pentominoes cannot be assembled into a 4×5 rectangle or why two can't be assembled into a 2×10 rectangle?

16. Use all 12 pentominoes to make three identical shapes of 20 squares each. (Each of the 12 pentominoes should be used exactly once in one of the shapes.)

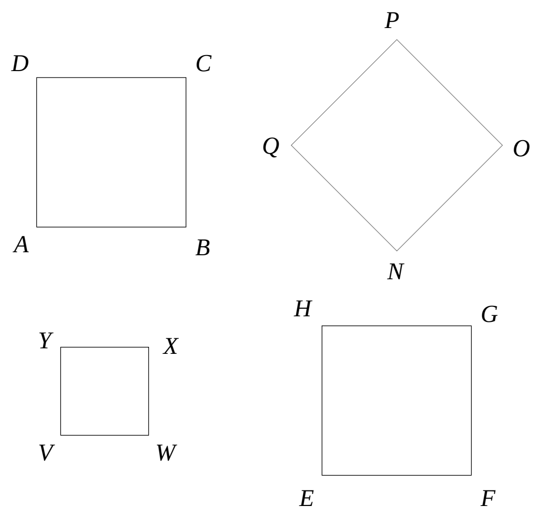
Investigation 2.2

Comparing Pictures

For Discussion

As a class, decide on what you will mean by "these two figures are the same."

To make a *wise* decision about what the statement should mean, you might consider some specific cases. For example, look at the four pictures shown here, and decide which ones you'd call "the same."



You will learn about another kind of sameness, called similarity, later in your studies of geometry.

You can think of the symbol as composed of two parts: $=$ and $\sim$. What aspects of "congruence"—same shape, same size—might these two parts stand for?

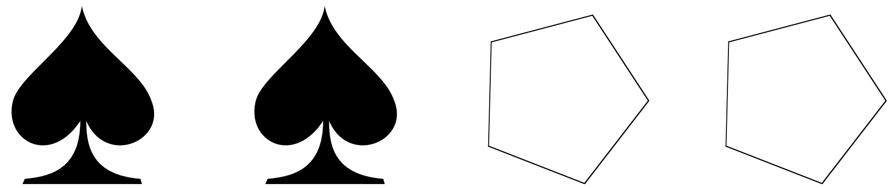
In some ways, *all* of the geometrical objects above are the same. (How?) And in some ways, they're all different. (How?) So, mathematical language has several words to distinguish "how much the same" two figures are. When two figures are "the same shape *and* size" (regardless of location or orientation), they are *congruent*.

Congruence is such an important mathematical relationship that it has its own symbol, $\cong$. The symbol is used this way:

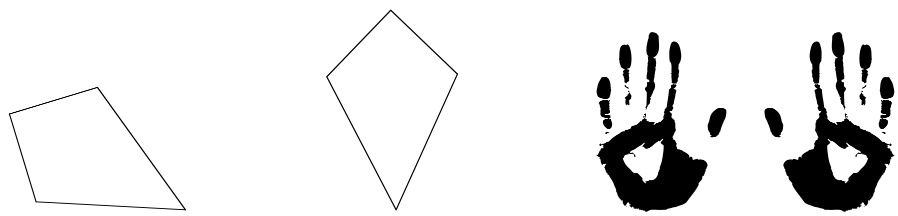
$$\square ABCD \cong \square EFGH.$$

COMPARING PICTURES (continued)

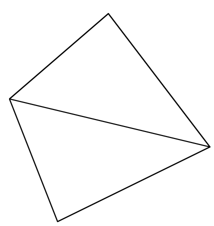
Pictures often suggest whether two shapes are congruent. Congruence is easiest to see when the pictures have the same orientation.

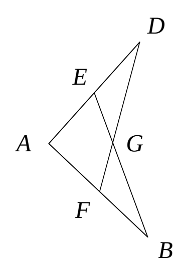


Congruence can be harder to see when the pictures are rotated or flipped.



It may be even harder when the shapes touch or overlap.

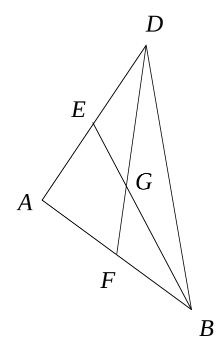




1. **"Find Your Partner"** Your teacher will give each student a sheet with some shapes on it. Your job is to find your "partner(s)"—students who received a set of shapes congruent to yours. It may take ingenuity to decide if two shapes really are congruent.

You may want to work with your partners on the rest of the problems in this section.

2. There are at least seven triangles in the figure below. It is easy to find some pairs of triangles that are *not* congruent. Name any pairs that you think are congruent.

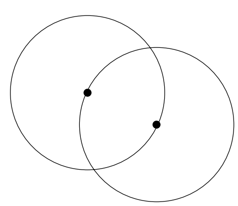


Using tracing paper or a cutout and measuring are all good strategies when you find yourself wanting some help deciding by "visual inspection" whether two figures are the same.

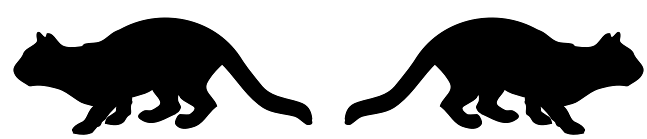
3. Look at the shapes below, and

a. decide whether the two shapes in each example "seem congruent" or "surely are not congruent";

b. list any measurements you could take, tests you could perform, or other checks you could make that would suggest whether the figures are congruent.

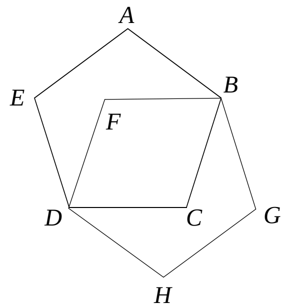


Two circles

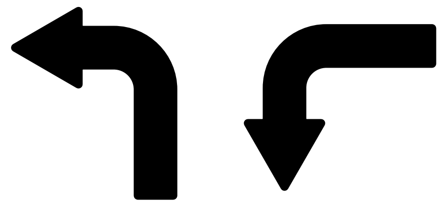


Two cats

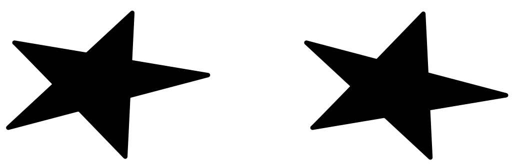
COMPARING PICTURES *(continued)*



Two pentagons: ABCDE and BGHDF



Two bent arrows



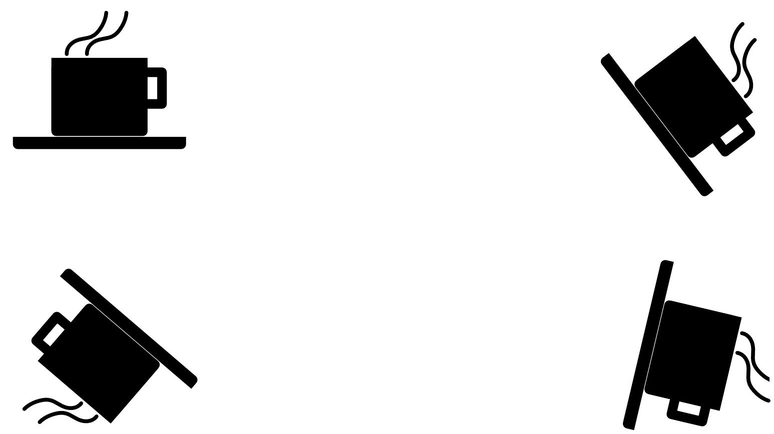
Two stars

Two snowflakes

Tests of “spatial reasoning” (or “spatial visualization”) abilities often present several pictures and ask which one is not like the others. In effect, these tests are asking about congruence.

4. For each set of four shapes, identify the figure that is not congruent to the other three and explain *how* it differs.

a.



b.

c.

WHAT IS CONGRUENCE?

When two figures are *congruent*, they have the same shape and size—but what *are* "shape" and "size"?

5. List any strategies you used in the previous problems to decide whether two look-alike shapes are congruent.

Because you will be talking about congruence with your teacher and classmates, you must come to an agreement with *them* on a definition to use. "Same shape, same size" isn't good enough unless you agree on what you mean by both expressions.

A good definition has just enough information—not too little and not too much.

6. Propose and write your own definition of congruence. Your definition should say what a person can *do* to decide if two figures are congruent.

FOR DISCUSSION

Discuss the definitions of congruence proposed by members of the class, and then construct one on which all of you can agree. Let this be your *working definition*—one that may change over time as your class learns more about the subject.

CHECKPOINT

7. Define the following words and symbols:

$\cong$

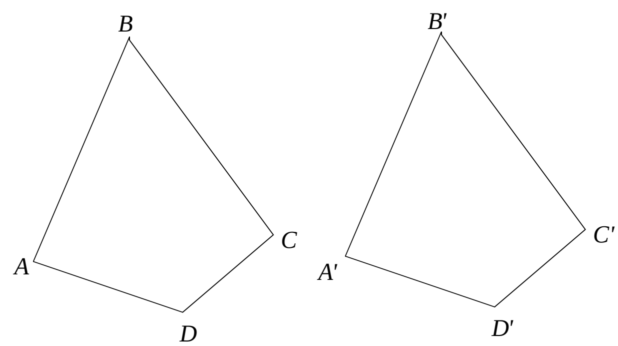
congruent

definition

square

8. Are all equilateral triangles congruent? Explain.

9. Are these two figures congruent? Justify your answer by using *your* definition of congruence.



TAKE IT FURTHER

10. What are the attributes of a "good definition"?

a. A dictionary is not always enough to tell you what a word means. Look up the following words in a dictionary and tell whether you think the dictionary does a good job of defining them.

triangle

puzzle

square

definition

the

geometry

b. If you found a definition you did not like, explain why.

Another problem with a dictionary is that you have to know how to spell a word *before* you look it up.

One problem with a dictionary is that it tells you what one word means by using other "simpler" words. But how do the writers know what's simpler to you?

You can use any word for this problem, not just mathematical words. Try an ordinary word like "simple" or "belong."

11. Pick a word and look it up in a dictionary. The definition will contain certain key words. Write these words down and then look *them* up. Keep doing this until you reach words with meanings that everyone in the class should agree upon. How many words are on your list? Did they keep getting "simpler"?

Investigation 2.3

THE CONGRUENCE RELATIONSHIP

You can have congruent line segments and congruent triangles, but you can also use the term "congruence" for figures that have more than just one or two dimensions. You can refer to congruent spheres and congruent tetrahedra, or congruent shapes in even higher dimensions.

1. **Write and Reflect** If you could use any tool or method you like, describe how you would decide if two line segments are congruent. What method would you use for angles? For triangles? What method would you use for two rectangular solids (boxes)? For cones? For cylinders?

The word *congruence* comes from the Latin word *congruens*, which means "to meet together." If you superimpose one figure on another, and they "meet together," then they are congruent.

One class decided on this test for congruence: Given two shapes drawn on paper, if you can cut one out and fit it exactly on top of the other shape (nothing hanging over on top or sticking out on bottom), then the two shapes are congruent. Their teacher supplied a name for the test: *superimposability*. Here is one way to say this:

Two figures are congruent if they differ only in position or orientation.

2. Will the cut-and-move test work for line segments?

3. How could you adapt the test to decide whether three-dimensional objects such as spheres or rectangular boxes are congruent?

LENGTH, MEASURE, AND CONGRUENCE

There are many other shape comparisons: *similarity* is one. There are also many other size comparisons. One is *perimeter*.

There are many ways of comparing figures. For polygons, you have been looking at a *shape and size* comparison: congruence. Later, you will look in detail at another size comparison: area.

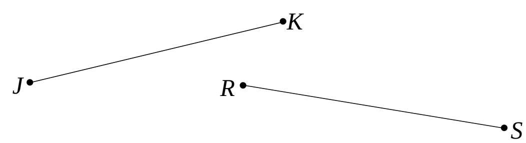
4. Use what you already know about area to answer the following questions:

   a. If two polygons are congruent, must they have the same area? Why or why not?

   b. If two polygons have the same area, must they be congruent? Why or why not?

A single geometric object—even a line segment—can have *many* different numbers attached to it, like length, slope, and location. So, when you are talking about a geometric figure, you must make it clear whether you are talking about the geometric

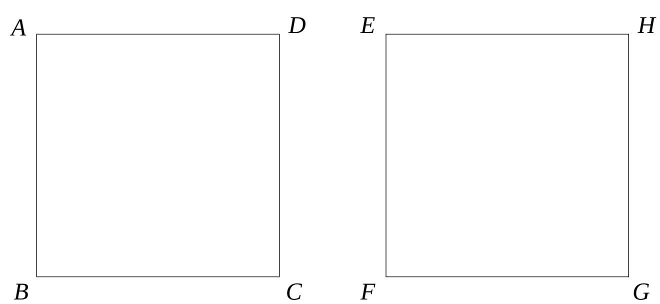
object itself (like a point or a segment or a circle) or about one of the many numerical values that help describe it.



Two segments: $\overline{JK}$ *and* $\overline{RS}$

Symbols have been designed to make this distinction clear. $\overline{JK}$ stands for a geometric object—the line segment joining points J and K. The symbol JK, without the "overbar," stands for the *length* of that segment, a *number*.

You can compare two shapes (are they congruent?) or two numbers (are they equal?), but it makes little sense to compare an object to a number. For example, here it looks like $\square ABCD \cong \square EFGH$.



But what could "$\square ABCD = 4$" mean? The perimeter is 4? The area is 4? The length of a side is 4? Certainly, the *shape* doesn't equal 4.

5. Here are four statements about line segments. For each one, state whether numbers, points, segments, or other objects are explicitly mentioned in the description.

a. $JK = RS$

b. $\overline{JK} \cong \overline{RS}$

c. $\overline{JK}$ and $\overline{RS}$ have the same length.

d. The distance from J to K is the same as the distance from R to S.

6. Here are two segments and some statements about them. All are grammatical sentences—they have the right overall form—but only some are correct. Others may be false, or they may mix up different ideas in a way that leaves them with no clear meaning at all. For each statement, say whether it is correct or not. If it is *not* correct, explain what's wrong.

English statements can *look* legitimate, but may not seem to say much when you examine them closely. Political propaganda is a good example. Linguist Noam Chomsky's clever sentence "Colorless green ideas sleep furiously" illustrates that good form (grammar) does not guarantee meaning.

a. $JK \cong RS$

b. $\overline{JK} \cong \overline{RS}$

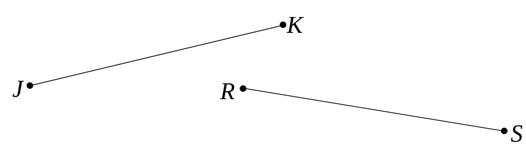
c. $JK = RS$

d. $\overline{JK} = \overline{RS}$

e. $JK = 1''$

f. $\overline{JK} = 1$

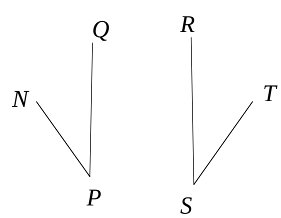
g. $\overline{JK} \cong 1''$



7. If two segments have the same length, are they congruent? Why?

8. If two segments are congruent, are they the same length? Why?

The symbols are different for angles, but the distinction is the same: an angle is a geometric object; its measure is a number.



$\angle NPQ \cong \angle RST$

When you mean the *angle* (the geometric object), write $\angle NPQ$ (or $\angle P$). When you mean the *size* or *measure* of an angle (what you find when you use a protractor), write $m\angle NPQ$.

In many geometry textbooks, angles are defined as being made up of two rays with a common endpoint. In *Connected Geometry*, we allow the sides of an angle to be either rays or segments. If we talk about the lengths of the sides of an angle, you can tell that we are considering the sides to be segments.

Notice that two angles can be congruent even when their sides have different lengths.

9. In what way is the sentence "$\angle NPQ = 56.6°$" confusing because of different meanings? What is the correct symbolic way to write "angle *NPQ* has a measure of 56.6 degrees"?

10. If $m\angle NPQ = m\angle RST$, are the two angles congruent? Why?

11. If $\angle NPQ \cong \angle RST$, do the two angles have the same measure? Why?

12. **Write and Reflect** Write descriptions of the methods you can use to tell if

a. two segments are congruent;

b. two angles are congruent;

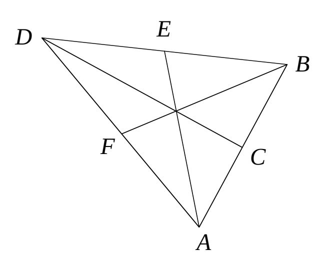
c. two triangles are congruent;

d. two cubes are congruent.

CHECKPOINT

The symbol $\perp$ means "is perpendicular to."

13. In $\triangle ABD$, $AD = BD$, $\overline{DC}$ is an altitude (that is, $\overline{DC} \perp \overline{AB}$), and F and E are midpoints. Judge each of the following statements as *true*, *false*, or *nonsensical*. Justify your answers. You may use measuring tools if you want.



a. $FD = DE$

b. $\overline{FD} = \overline{DE}$

c. $\overline{FD} = 1.5$ cm

d. $\angle ACD = 90^\circ$

e. $\triangle DFB = \angle DEA$

f. $\angle ACD$ is a right angle.

g. $\overline{FA} \cong \overline{BE}$

h. $\overline{FA} \cong \overline{BD}$

i. $\angle ADC = \angle BDC$

j. $m\angle ADC = m\angle BDC$

k. $m\angle DFB \cong m\angle DEA$

l. $\angle DFB \cong \angle DEA$

m. $\triangle DCA \cong \angle DCB$

n. $\triangle DCA \cong \angle EAD$

Remember, two things are congruent if they differ only in their position or orientation.

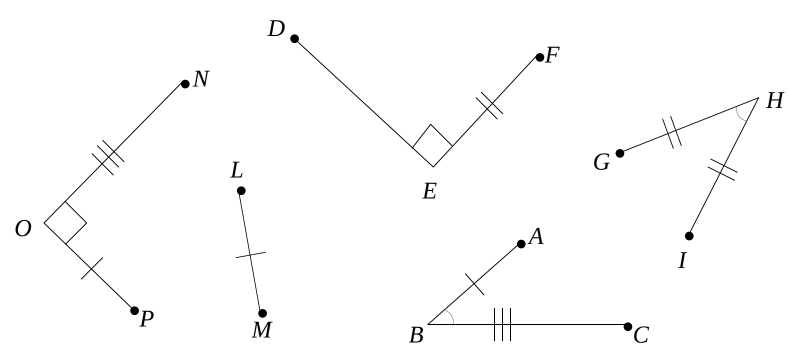
14. Is the following statement true or false? Justify your answer.

If two triangles are each congruent to the same triangle, then they are congruent to each other.

TICK MARKS AND OTHER SYMBOLS

You may want to use some tools to make a few measurements.

15. The picture below shows nine segments. Four pairs of segments are joined, showing four angles. The segments and angles are specially marked. Explain what meaning you think the marks may have.



For example, if you think that $\overline{AB}$ is congruent to $\overline{LM}$, write $\overline{AB} \cong \overline{LM}$.

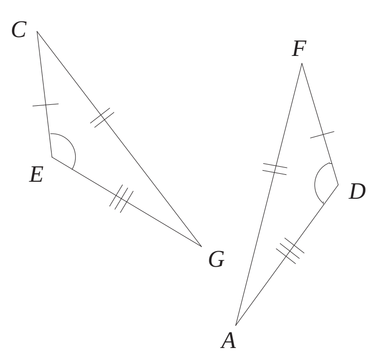
When drawing by hand, without any tools, it's hard to draw identical shapes. And sometimes you don't have tools, or don't want to take the time to measure drawings. So the tick marks can be really useful.

If segments have *different* markings, does that mean that they *are not* congruent?

16. Test the segments and angles for congruence and then write congruence statements using the proper symbols. (Do your statements relate to your theory about the tick marks?)

Symbols and pictures help clarify mathematical communication. For example, tick marks like the ones shown on the previous page indicate that line segments and angles are intended to be congruent. Even if someone's drawing isn't exact, identical tick marks on segments or angles will signify congruence.

CORRESPONDING PARTS



17. The two triangles shown here are congruent, yet only *one* of the congruence statements below is considered correct. Come up with your own reason for choosing one as "better" than all the others, and explain your reason.

a. $\triangle DFA \cong \triangle GCE$

b. $\triangle DFA \cong \triangle EGC$

c. $\triangle DFA \cong \triangle CEG$

d. $\triangle DFA \cong \triangle ECG$

e. $\triangle DFA \cong \triangle GEC$

f. $\triangle DFA \cong \triangle CGE$

18. Even though only one of the *above* statements was correct, there are other correct congruence statements for these two triangles. How many? Write them.

Ways to Think About It

The order of vertices is important in statements of congruence because it allows people to communicate lots of information with just a little bit of writing. When you read the sentence $\triangle QRS \cong \triangle XYZ$, you learn that those two triangles are congruent. But you also find out that

- $\overline{QR} \cong \overline{XY}$
- $\overline{RS} \cong \overline{YZ}$
- $\overline{QS} \cong \overline{XZ}$
- $\angle Q \cong \angle X$
- $\angle R \cong \angle Y$
- $\angle S \cong \angle Z$.

A good notation system—in mathematics, in music, in anything—should be efficient.

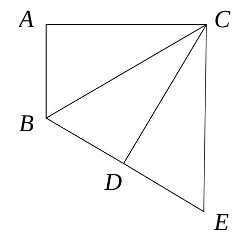
There's a lot of information packed into that one small sentence! The mathematical way to describe this matching of parts in a congruence statement is, "corresponding parts of congruent figures are congruent."

19. Imagine two congruent pentagons that are not regular. How many pairs of corresponding parts do they have? Sketch your pentagons and write a congruence statement.

20. Draw, label, and tick mark your own example of two differently-oriented congruent triangles. Exchange them with a partner, and write the appropriate congruence statements for corresponding parts.

21. **Write and Reflect** Explain in your own words—with pictures, if you like—what "corresponding parts of congruent figures are congruent" means.

22. The figure below contains three congruent triangles.



a. Name the three congruent triangles.

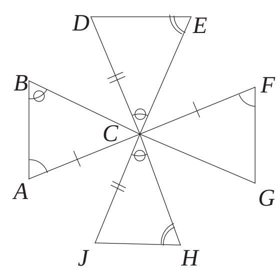
b. On your own sketch, mark congruent corresponding parts.

What is the mathematical definition of a *kite*?

c. Shape $ABDC$ in the figure is called a *kite*. In that kite, name the triangle that is congruent to $\triangle ABC$.

d. In $\triangle BCE$, name the triangle congruent to $\triangle ECD$.

23. The figure below is not drawn to scale, but all parts marked as congruent are *intended* to be congruent.



If $m\angle F = 80^\circ$, $m\angle H = 50^\circ$, and $m\angle B = 40^\circ$, then . . .

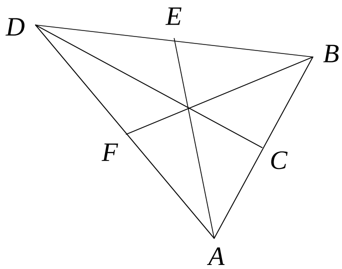
a. What are the measures of $\angle D$, $\angle E$, and $\angle A$?

b. Redraw the figure. Draw all angles with the correct degree measure, and make all congruent segments actually congruent.

Investigation 2.4

STRONG LANGUAGE

The original problem said "In $\triangle ABD$, $AD = BD$, $\overline{DC}$ is an altitude (that is, $\overline{DC} \perp \overline{AB}$), and . . . "



Earlier, you were given this picture and some information about it. You were then asked whether various statements about the figure were true. Suppose that you had been asked if the statement "$\triangle DCA \cong \triangle DCB$" is true.

In *this* picture, you could tell by measuring, or by cutting one triangle out and seeing if it fit on the other. But what if the picture had been drawn badly, or not drawn at all?

One purpose of mathematics is to allow you to look beyond unimportant details and understand the big picture.

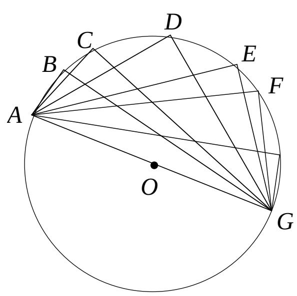
Unimportant details: It hardly matters, in *this particular* drawing, whether $\triangle DCA \cong \triangle DCB$.

The big picture: A *strong* statement would be "if *any* triangle (call it $\triangle ABD$) is isosceles ($AD = BD$), then an altitude to the third side ($\overline{DC} \perp \overline{AB}$) divides the triangle into two congruent triangles ($\triangle DCA \cong \triangle DCB$)."

FOR DISCUSSION

Discussion A: Suppose that the strong statement above happened to be true. Can you verify such a strong statement using only measurement or superimposition?

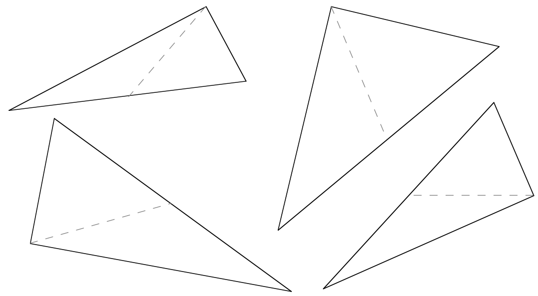
Discussion B: This picture illustrates what is meant by "angles inscribed in a semicircle." Below are two statements about angles inscribed in a semicircle. Which is the stronger statement? If you could make perfectly precise measurements, which statement could you verify with measurement alone?



Statement 1: $\angle ABG$, $\angle ACG$, $\angle ADG$, $\angle AEG$, and $\angle AFG$ in the picture are "angles inscribed in a semicircle," and all are right angles.

Statement 2: Any angle inscribed in a semicircle is a right angle.

1. The picture below shows four right triangles, each with a segment (dashed) connecting the right angle vertex with the midpoint of the hypotenuse.



Statement 1: In the pictures here, the length of the dashed segment in each triangle is half the length of the hypotenuse of that triangle.

Statement 2: The midpoint of the hypotenuse of a right triangle is equidistant from its three vertices.

Which statement is stronger? Which could be verified by (perfectly precise) measurement? Explain.

Investigation 2.5

TRIANGLE CONGRUENCE

Learning about triangles can teach you a lot about geometry in general.

Why is so much attention given to triangles and their properties? One reason is that any polygon can be cut up into triangles, so properties of other polygons often can be derived by thinking about the triangles from which they're made.

This investigation will focus on deciding whether two triangles are congruent. Up to now, the methods we've used have been measurement or superimposition—seeing if one could fit exactly on top of the other. But these methods can't always be used and don't help us confirm strong statements.

. . . powerful but long to write. It is often abbreviated as CPCTC.

How do you form the *converse* of a statement?

A more powerful tool is "corresponding parts of congruent triangles are congruent." It says that if you know that two triangles are congruent, then you also know that six pairs of parts are congruent—three pairs of sides and three pairs of angles. This can help us find congruent triangles if the "converse" holds as well. That is, maybe if the six pairs are congruent, side-for-side and angle-for-angle, then the *triangles* will be congruent. In fact, this is true. The following statement can be a congruence test:

> *If you can match up the sides and angles of two triangles in a way that all six pairs of corresponding parts are congruent, then the triangles are congruent.*

FOR DISCUSSION

If this is to be a test for congruence, it had better not conflict with the idea that "two triangles are congruent if they differ only in their positions or orientations." Does it conflict? Explain.

1. In two ways, show that a diagonal divides a square into two congruent triangles.

2. Does the diagonal of a rectangle divide the rectangle into two congruent triangles? What about the diagonals of parallelograms? Trapezoids? Explain your answers.

HOW MUCH IS ENOUGH?

Showing that two things are the same by showing that they share so many properties that they *must* be the same is a common technique in mathematics. Some people say it this way: "If it walks like a duck, looks like a duck, and quacks like a duck, then it must be a duck."

To show that two triangles are congruent, one might show that they share so many properties that they couldn't differ except in position or orientation. For example, if two triangles have sides and angles measuring 3″, 90°, 4″, 37°, 5″, and 53° (around each triangle in that order), then all of their parts are congruent. The triangles themselves *must* be congruent.

FOR DISCUSSION

Discussion A: Could you save time by checking *fewer* than all six pairs of corresponding parts for congruence and still be guaranteed that two triangles are congruent? For example, if you know that all three angles in one triangle are congruent (angle-by-angle) to all three angles in another, can you be sure the triangles are congruent? What if three pairs of sides are congruent?

Discussion B: "Information that is enough to specify one triangle exactly is also enough to ensure that two triangles are congruent." Do you believe this statement is true? Give reasons.

THE ENVELOPE GAME

As long as you can make *more* than one triangle with the information you have, keep taking notes out of the envelope. You are looking for enough pieces to *determine* a triangle.

Your teacher will give you an envelope containing six pieces of information about a triangle—three sidelengths and three angle measures—on separate notes. Take out one note at a time until you have *just* enough information to build *only one* triangle with exactly those measurements. You can use any construction tools or techniques you want: compass, ruler, protractor, geometry software, paper folding, ...

When you are done, the triangle you have made should agree with all the information you pulled out of the envelope and all the information that's still in the envelope. So check the remaining notes to be sure that you made the correct triangle.

3. How many pieces of information did you need? Draw your triangle, and list the information you used.

FOR DISCUSSION

Did everyone in your group or your class need the same number of pieces of information? If some groups needed fewer pieces than others, explain why.

How Unlucky Can You Get? Depending on the luck of the draw, different students may need different numbers of pieces of information before they can "nail down" the triangle.

For example, would you ever need all six?

4. What is the largest number of notes you'd ever have to draw before you could construct the triangle? Give reasons for your answer.

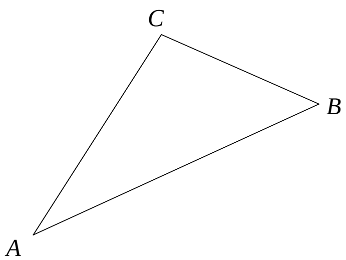
Is one note ever enough?

5. What is the smallest number of notes you could draw and still construct exactly one triangle that uses all the information? Will *any* collection of that many notes *always* be enough?

FOR DISCUSSION

Summarize what you've found. There seems to be a "magic number": fewer pieces of information always allow you to build more than one triangle, and more pieces of information means that there's only one possible triangle you can build. However, if you have that "magic number" of pieces, you may have exactly one triangle or you may still have more than one, depending on *which pieces* of information you have. What is the magic number? Which sets having that many pieces of information allow for only one triangle to be built?

CLASSIFYING YOUR INFORMATION



The six pieces of information about a $\triangle ABC$ are the lengths of $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ and the measures of $\angle A$, $\angle B$, and $\angle C$.

6. There are many *combinations of three* pieces of information about the triangle here. List all the possible combinations. The list is started below:

- $m\angle A$, $m\angle B$, $m\angle C$
- AB, $m\angle A$, $m\angle B$

Your list from Problem 6 would be a good code to use if all triangles were named *ABC*. The code given here is designed to work for *all* triangles no matter how they are labeled.

Your combinations could be classified in many ways. Here's a scheme that people have found useful, and a set of abbreviations—a kind of code—to help remember the scheme.

Three angles: There's one combination of three angles.

AAA: The code uses "A" to stand for "angle."

Two angles and a side: Here you have two arrangements.

"ASA" should be read "angle-side-angle."

Order is important; ASA is not the same as AAS.

ASA: This symbol means that the side you know is between the two angles you know (for example, $m\angle A$, $m\angle B$ and AB).

AAS: This means you know two angles and a side that's *not* between them (for example, $m\angle A$, $m\angle B$, and BC).

Two sides and an angle: Two arrangements again:

SAS: You know two sides and the angle between them (for example, AB, BC, and $m\angle B$).

Again, the order will turn out to be important. SAS is not the same as SSA.

SSA: You know two sides and an angle *not* between them (for example, AB, BC, and $m\angle C$). This abbreviation is used no matter which of the *non-included* angles you're talking about. (Think about why.)

Three sides: Only one way:

SSS: You know all three sides.

A NEW ENVELOPE GAME

You will again get an envelope, but this time it will contain only three measurements. Different envelopes will contain different combinations of three pieces of information, so your class can compare them.

7. As in Problem 3, the goal is to construct a triangle from the clues in your envelope. Call your set of clues “good” if it lets you make *exactly* one triangle. Call the set “bad” if it doesn’t fit any triangle, or if it fits more than one.

- **a.** Write down your clues and classify them by one of the three-letter codes (SAS, AAA, and so on).
- **b.** Decide whether the set of clues is “good” or “bad.” If it’s good, show the triangle. If it’s “bad,” show why.

FOR DISCUSSION

Discussion A: Compare results with others in your class. Which three-letter codes are “good” clue sets?

Discussion B: List the “good” sets of measurements—the ones that uniquely determine a triangle. We’ll call these the “Triangle Congruence Postulates.” Beside each postulate, write what it says and illustrate it with a picture.

Information that is enough to specify one triangle exactly is also enough to ensure that two triangles are congruent.

The fact that these are *postulates* is important. You have not proved that they work, but you have strong evidence for it from the envelope game.

Whenever you set out to prove things, you need to work with some assumptions, called *postulates* or *axioms*. These are things you are convinced are true, and they give you a starting point to build proofs. In the rest of this module, you will be writing proofs of theorems based on these postulates. But these theorems are only “true” as long as we believe the postulates.

WAYS TO THINK ABOUT IT

The triangle congruence postulates say that if *two* triangles share certain pieces of information (for example, SSS), then they are congruent. They also say that it's possible to build *only one* triangle from these three pieces of information. Well, isn't there a contradiction? Are there *two* triangles or *only one*?

The answer is connected with what is meant by "one," and that is connected to the discussion on page 4 in Investigation 2.2 about the meaning of "the same." When people say "Given three sides, only *one* triangle can be built," they *really* mean that any *two* triangles built from these sides must be congruent.

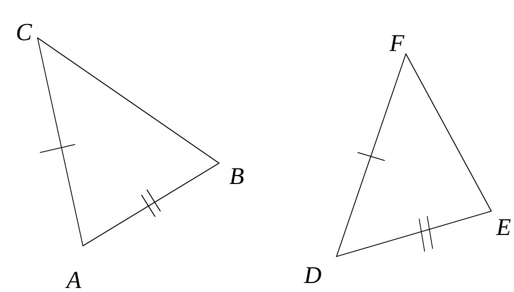
CHECKPOINT

For Problems 8–13:

a. Tell if the given information is enough to conclude that the triangles are congruent.

b. If so, state the triangle congruence postulate that can be applied, and describe a way to line up the vertices to show that the triangles are congruent.

8.



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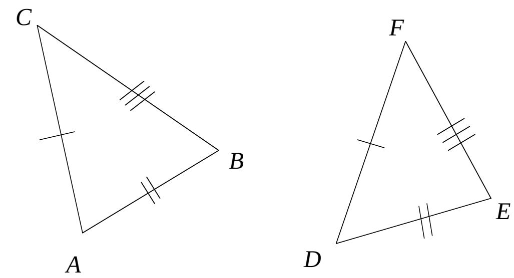

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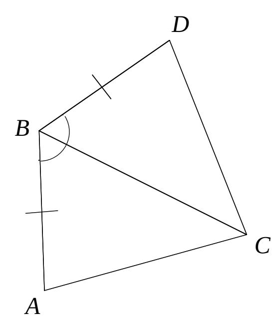

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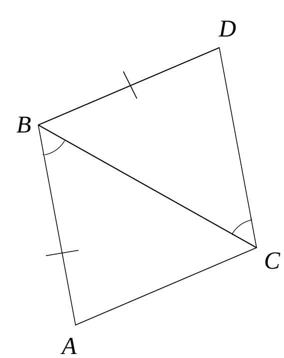

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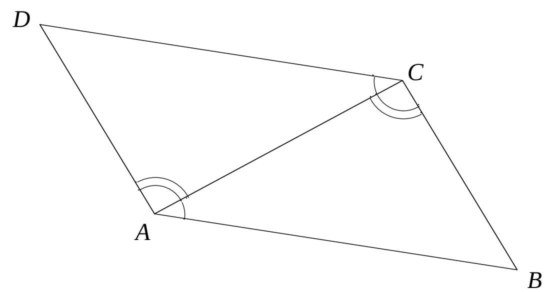

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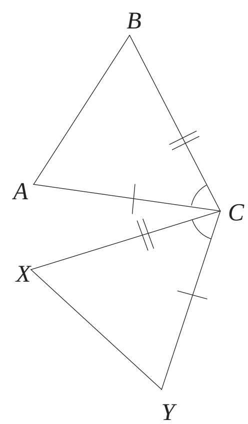


COUNTEREXAMPLES

What does *counterexample* mean? What other words use *counter-* or *contra-* in the same way?

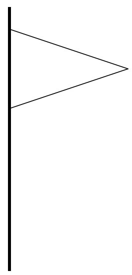
***Does* an altitude to the "third" side of an isosceles triangle always do this? Can you write a proof for that fact right now?**

Later on, you will take an in-depth look at how to use the triangle congruence postulates to prove strong statements like "in *any* triangle with two congruent sides, the altitude to the third side divides the triangle into congruent triangles."

In *disproving* a statement, however, all you need is *one* counterexample: one case for which the statement doesn't hold. The conjecture may still ring true for a few, or even many other examples; but it is not a true statement until it is formulated in such a way that it is true in every possible case.

In the following problems, you are asked to show that the given information is not enough to define only one triangle. Your counterexamples will be two triangles that both fit the given information.

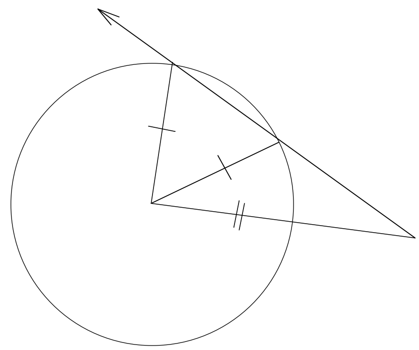
Your friend hasn't studied geometry yet, so it isn't safe to assume your friend means "SAS" when you discover the message isn't stated clearly.



14. You and a friend are making triangular banners (pennants) for a school project and you want them all the same size and shape. But you're not there when your friend calls you with the design. The message doesn't give you enough information. It states that the design has a 30° angle at the tip of the pennant, a 14-inch side, and an 8-inch side. Show why this is not enough information.

Why do you think the author found it convenient to draw a circle?

15. The following picture is a "proof without words" that two congruent corresponding sides and a congruent non-included angle (SSA) is not enough to guarantee that two triangles are congruent. Add the words (and labels, if you want) that explain the proof.

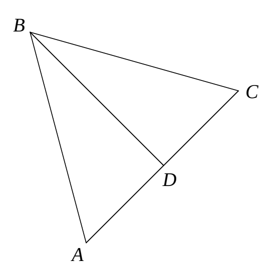


16. Find a counterexample to this statement: "AAA (all three angles have equal measure) is sufficient information to prove congruence in two triangles." Explain why it disproves the conjecture. Include a picture in your explanation.

17. **Write and Reflect** If you want to prove that something is true about all triangles, all circles, all numbers, and so on, thousands of examples won't prove your conjecture. Yet only one counterexample will disprove it. Explain.

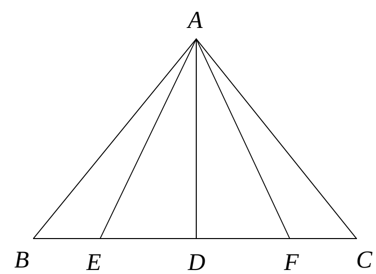
What does *perpendicular bisector* mean?

18. In the figure below, $\overline{BD}$ is the perpendicular bisector of $\overline{AC}$. With this information, two triangles can be proved congruent. Name them. Prove that they are congruent.



TAKE IT FURTHER

19. In the figure below, $\overline{AD}$ is the perpendicular bisector of $\overline{BC}$. With this information, one pair of triangles can be proved congruent. (You did this in Problem 18.)



Adding just one more piece of information might make it possible to prove more triangles congruent. For each example below, decide whether the additional information makes that possible. If you think it does, name the new congruent triangles and state how you have proved them congruent (SSS, SAS, and so on).

a. What if $AB = AC$?

b. What if $\overline{AD}$ is also the perpendicular bisector of $\overline{EF}$?

c. What if $\angle EAD$ is congruent to $\angle FAD$?

20. In a regular polygon, if all the diagonals from *one* vertex are drawn, will they divide the polygon into triangles that are all congruent to each other? Explain.

21. In a regular polygon, if *all* the diagonals are drawn, will every polygon formed have at least one other matching congruent polygon?

22. Study the two Logo procedures below. Will the figures drawn be congruent?

a. `forward 20 back 20 right 150 forward 20`

b. `forward 20 left 30 forward 20`

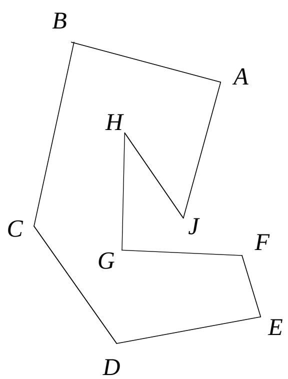
23. Imagine two parallel planes intersecting a cube. Is it possible for the cross sections to be congruent? Explain.

24. Draw a triangle on the coordinate plane. Call it $\triangle DEF$. Write the coordinates of points D, E, and F.

Example: if *D* is the point (1, 4), adding 2 to both the *x*- and the *y*-coordinates would give a new *D* at (3, 6).

a. If you make a new triangle by taking points D, E, and F and adding 2 to every coordinate, will the new $\triangle DEF$ be congruent to the original $\triangle DEF$?

b. If you make another new triangle by taking the original points D, E, and F and multiplying every coordinate by 2, will the new $\triangle DEF$ be congruent to the original $\triangle DEF$?



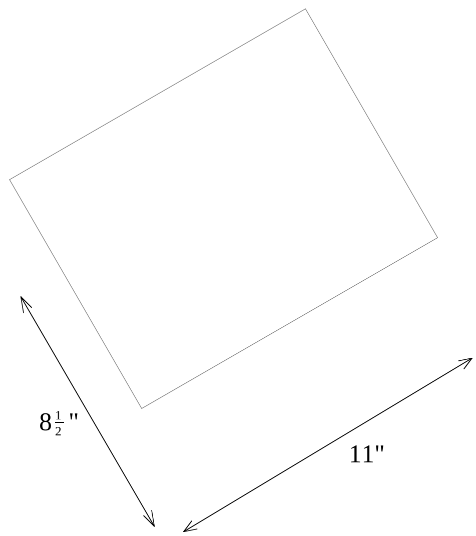
25. Cutting Polygons into Triangles At the beginning of this investigation, you read that "any polygon can be cut up into triangles." But it's not so easy to say how to do this. For example, in some polygons, one could "pick any vertex and draw all the diagonals from that vertex." But that rule would not work for the crazy polygon *ABCDEFGHJ* shown here. Can you find a method that will *always* work? Or is the statement, perhaps, not really true of *all* polygons?

EXTENSION: CUTTING UP CONGRUENTLY

Each of the following problems asks you to cut a particular shape into some number of congruent pieces. First, trace the figure onto another piece of paper. Then, either cut your copy into congruent pieces or draw lines to show how such cuts could be made.

Which method makes it easier to check for congruence?

26. Use a whole piece of paper as your rectangle. (You will need several sheets of paper.)



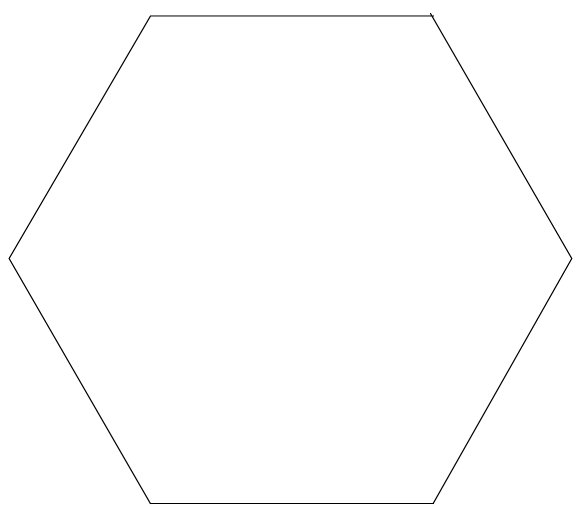
a. Divide your rectangle into two congruent pieces.

b. Divide another rectangle into three congruent pieces.

c. . . . four congruent pieces.

d. . . . six congruent pieces.

27. Make five copies of the hexagon below.



a. Divide one copy into two congruent pieces.

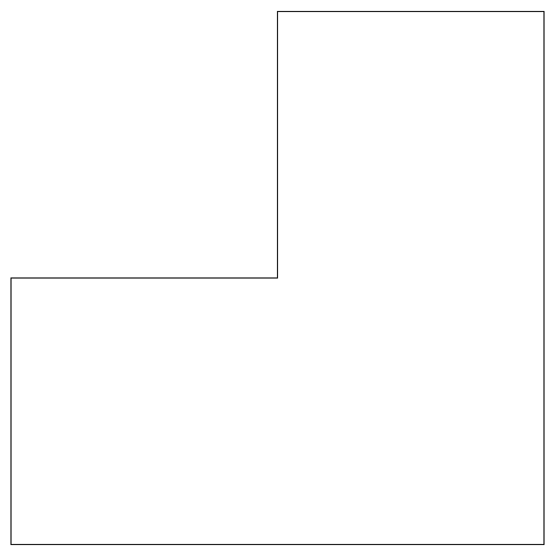
b. Divide another into three congruent pieces.

c. . . . four congruent pieces.

d. . . . six congruent pieces.

e. . . . eight congruent pieces.

28. Make four copies of the shape below.



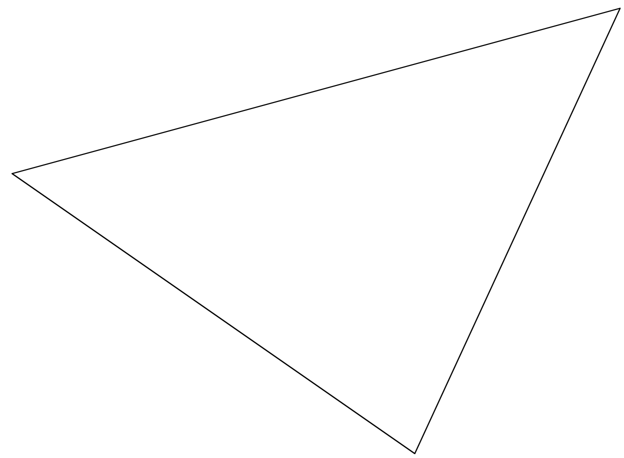
a. Divide one copy into two congruent pieces.

b. Divide another into three congruent pieces.

c. . . . four congruent pieces.

d. . . . six congruent pieces.

29. Trace the triangle below.



a. Divide it into two congruent pieces.

b. Divide it into four congruent pieces.

c. Could you divide *any* triangle into two congruent pieces? Four congruent pieces? Try to justify your answer.

In this case, "Justify your answer" means that you should provide an explanation if your answer is yes, or provide a counterexample if your answer is no.

30. **Project** Draw your own triangle, then divide it into some number of congruent pieces; for example, 2, 3, 4, or 5 pieces. Which numbers of congruent pieces can you *always* construct, no matter what kind of triangle you use? Which numbers require special triangles? Are there any that are impossible? Write about why you think some numbers require special triangles, and why some, if any, are impossible.

31. **Project** Take a square and divide it into two congruent triangles. In fact, try to divide it into 3, 4, 5, and 6 congruent triangles. For what numbers n is it easy to divide your square into n congruent triangles? For what numbers n is it difficult? For what numbers n is it impossible? Explain.

Investigation 2.6

WARM-UPS FOR PROOF

WHAT IS A PROOF?

A proof in mathematics is a logical argument designed to explain a new observation in terms of facts one already understands. In a proof, you start with statements that everyone agrees with. From them, using only logic that everyone agrees with, you convince yourself (or others) that other statements *have* to follow. It must be clear that no alternate interpretations or counterexamples are possible.

What does a proof look like? This is trickier. In mathematics, not all proofs look the same because there are many different ways to make a convincing argument. A proof could include

- a picture or set of pictures,
- a set of equations,
- an essay,
- a series of statements accompanied by justifications,
- a construction using compass and straightedge or computer drawing tools,
- a computer program.

In the following investigations, you will learn how to come up with mathematical proofs and also how to write up your proofs.

You probably have found yourself in a situation where you wanted to convince other people of some fact. The "fact" may be a point of view that you hold (like "swimming is more fun than soccer"), or it may be something more objective (like "smoking cigarettes causes health problems"). In the latter case, you may use arguments that are close to mathematical *proof*.

1. In the second week of school, a P.E. teacher sends you a note saying that you've been absent from her third period P.E. class. *Your* schedule says that you have geometry third period, and P.E. fifth period (with another teacher), and that's what you've been following. How would you convince the teacher that *her* schedule is the one that is wrong?

How is the word "argument" used differently in mathematics than in everyday language? What is the etymology (derivation) of the word "argument"?

2. Divide a square into four pieces each having the same area. Write an argument that will convince a friend that the four pieces really do have the same area. Try it out, and see if the argument is convincing.

3. Divide a square into five pieces each having the same area. Again, write an argument and check to be sure that it is convincing.

4. What kind of number, odd or even, do you get if you add *any* odd number to *any* even number? Write two convincing arguments for this fact.

- **a.** Write one argument that would convince someone in the fourth grade.
- **b.** Write another argument that would be understood by someone who knows algebra.

5. Suppose that you know that the sum of the measures of the angles in a triangle is 180°. Use this fact to show that the sum of the measures of the angles in a quadrilateral is 360°.

Investigation 2.7

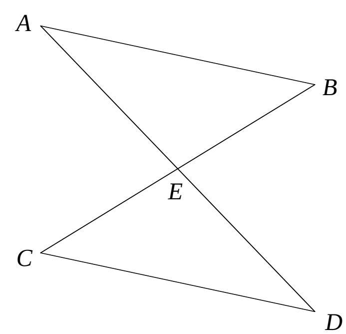
Writing Proofs

As you might imagine, the way a proof is expressed—the way you make your argument—can vary quite a bit. The *logic* of a proof is determined completely by mathematics and rules of reasoning, but the *look* of a proof is part of the customs and culture of the people who are its audience. For high school students, this depends to some extent on the country in which you study. Schools in China, Israel, France, or Russia sometimes teach ways to present proofs that are quite different from those taught in most American schools.

Below you will find one simple proof and several different correct ways to write it. Read with two goals:

- to compare the different methods of writing;
- to pull from all of these proofs what they have in common—the essential elements of proving triangles congruent.

The Claim: Given that $\overline{AB}$ and $\overline{CD}$ are parallel and that E is the midpoint of $\overline{AD}$, prove that $\triangle ABE$ is congruent to $\triangle DCE$.



Styles of Proof

Paragraph Proof

In this style, a series of statements fit together logically to establish that the two triangles are congruent. The statements are written in paragraph form.

> Because $\overline{AB} \parallel \overline{CD}$, the alternate interior angles are congruent, so $\angle ABE \cong \angle DCE$ and $\angle BAE \cong \angle CDE$. Also, because E is the midpoint of $\overline{AD}$, then $\overline{AE} \cong \overline{ED}$. Therefore, by AAS, triangles ABE and DCE are congruent.

TWO-COLUMN STATEMENT-REASON PROOF

A set of statements is written in the left column. Each statement has a reason why it must be true: that reason is written in the right column. The last statement is what is being proved. The reasons are postulates, theorems, definitions, or "givens" (statements that everyone assumes for this problem).

Statements	Reasons
1. $\overline{AB} \parallel \overline{CD}$	1. Given
2. $\angle ABE \cong \angle DCE$	2. Parallel lines form congruent alternate interior angles with a transversal.
3. $\angle BAE \cong \angle CDE$	3. Parallel lines form congruent alternate interior angles with a transversal.
4. E is the midpoint of $\overline{AD}$.	4. Given
5. $\overline{AE} \cong \overline{DE}$	5. The midpoint is defined as the point that divides a segment into two congruent parts.
6. $\triangle ABE \cong \triangle DCE$	6. AAS

OUTLINE STYLES

Some students who studied in China use the following form. The symbol $\because$ means "because," and is used for information that is given. The symbol $\therefore$ means "therefore." Some important reasons are included in parentheses.

$\because \overline{AB}$ is parallel to $\overline{CD}$.

$\therefore \angle BAE \cong \angle CDE$

$\therefore \angle ABE \cong \angle DCE$ (alternate interior angles)

$\because E$ is midpoint of $\overline{AD}$.

$\therefore AE = DE$

$\therefore \triangle ABE \cong \triangle DCE$ (AAS)

Another Outline Style This style was used by some students who studied geometry in Russia. A statement is written and justifications for it are recorded below in outline form.

Given that $\overline{AB}$ is parallel to $\overline{CD}$ and E is the midpoint of $\overline{AD}$,

$\triangle ABE \cong \triangle DCE$ by the AAS postulate because

- $\angle BAE \cong \angle CDE$ (alternate interior angles);
- $\angle ABE \cong \angle DCE$ (alternate interior angles);
- $AE = DE$ (E is midpoint of $\overline{AD}$).

1. **Write and Reflect** All four proofs show that triangle ABE is congruent to triangle DCE using the AAS triangle congruence postulate. What else do all these proofs have in common? What are the advantages and disadvantages of the different styles?

WHY PROOF?

In the previous section, you saw four different ways to present a proof. But that might be getting ahead of the story. A more basic question is, "Why bother with proof at all?" A surprisingly complex answer comes from a mix of tradition, necessity, and culture.

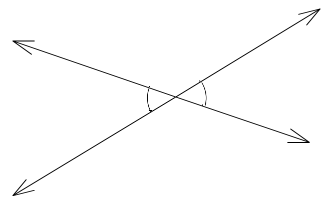
If you ask a mathematician what makes mathematics different from other disciplines, you'll probably hear something like, "Mathematicians prove things."

Mathematicians have always been experimenters: they perform thought experiments, build and experiment with models, and gather data. From the data they make conjectures, but for conclusions, they must rely on *deduction* and *proof*. New results come from reasoning about things that *must follow logically* from what is already known or assumed. The mixing of deduction and experiment is one of the distinguishing features of mathematical research.

Deduction and experiment are different *habits of mind*.

Deduction and experiment are different ways to think about things. As an example of how they are different, consider the following conjecture:

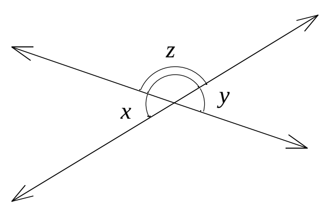
> If two lines intersect, the angles that aren't adjacent to each other (the so-called *vertical angles*) have the same measure.



Geometry software is ideal for this experiment. So is paper folding.

2. Experiment Set up an experiment to test the conjecture.

3. Deduction Write a deductive proof of the fact that vertical angles always have the same measure.



What's the relation of x to z? Of y to z?

4. Write and Reflect Compare the methods used in Problems 2 and 3. Which is more convincing to you? What are the advantages of each? Disadvantages?

What kinds of problems might have been important to early scientists and mathematicians?

5. One of the reasons that the deductive method became so important in mathematics is that there are many mathematical problems for which it is extremely difficult (or even impossible) to perform a complete experiment. Give an example of a mathematical problem for which deduction would be the only feasible approach.

6. Consider the following conjecture:

> If you have five points, not all on a straight line, then it is possible to find two of them so that the line (not just the line segment) through these two misses the rest.

Write about how you would investigate this conjecture. What are the advantages and disadvantages of deduction and experiment in trying to establish this conjecture?

PERSPECTIVE ON DEDUCTION AND EXPERIMENTATION

In modern mathematics, there are several important conjectures that are widely believed to be true on the basis of experimental evidence, but can't be used as a foundation for new ideas because no one has yet been able to prove them.

Deduction and experiment aren't the only ways people construct new mathematics. Srinivasa Ramanujan (1887–1920) was an Indian mathematician who came up with very deep mathematical results via a process that few people understand even today. For more details, see [2].

Note: Numbers in brackets as above refer to the Bibliography found at the end of this essay.

When mathematicians work on problems, they almost always use a mixture of deduction and experiment. But a tradition grew up in Greek mathematics around 300 B.C. that survives even today:

> *A conclusion is not accepted in mathematics unless someone can produce a deductive proof for it.*

Why is such rock-solid assurance so important? Mathematics builds new ideas by fitting together older ideas in new ways. The new ideas will not stand up if their foundations—the older ideas on which they are based—are not solid.

Some people feel that one disadvantage of this tradition is that while mathematical results are presented as a beautiful flow of deduction, they are almost never *obtained* this way. The presentation hides the author's research process. You (the reader) see the results, but not how they were discovered. You don't get to see the false starts, the special cases, the reasons for taking certain steps, or the sudden insight that caused everything to fall into place. High school textbooks have also tended to follow this tradition, with the result that many show just the facts, procedures, and perhaps proofs, but not how they were discovered or invented. Hiding the human processes that created our mathematics can make it seem mysterious and forbidding, and can give the impression that memory, rather than reasoning, is the key to success in mathematics. There is also a difficulty in maintaining the tradition of deductive proof. Mathematics is getting to be a huge enterprise. The number of new results published each year is quite large, and each new result is based on the validity of other research papers. No one can check all this, and there are many examples of incorrect results that are published, complete with alleged proofs that contain subtle errors.

Still, the deductive tradition for presenting mathematics has been responsible for much of the success and the appeal of the subject over the past 2000 years, and there's little evidence that this tradition will change any time soon.

Bourbaki was a pseudonym for a group of French mathematicians that formed after World War II. They describe their work as a series that "takes up mathematics at the beginning, and gives complete proofs." For a discussion of the Bourbaki collaborative, see [1, 3]. In fact, it is not certain whether the famous Euclid was a person or a pseudonym for a group of people (see [3]).

Over the centuries, there have been several attempts to give deductive treatments for whole areas of mathematics. The idea is to state a few simple axioms and then to deduce from these axioms everything that is currently known about some field. Two very famous and influential works of this type are Euclid's *The Elements* (300 B.C.) and Bourbaki's *Elements of Mathematics* (a project that was begun in the middle of this century and continues today).

7. **Write and Reflect** Research Ramanujan [2], Euclid [3], or Bourbaki [1, 3]. Write a report or make a presentation on the style of mathematics (both research and reporting) that your person (or group) used.

8. A mathematician who consulted with the authors of this module said that she hates TV ads because they insult people's intelligence by making claims and not offering a bit of evidence to support the claims. Write about some experience or situation in which *you* were especially bothered by people not justifying what they said.

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A Beginner's Manual

Problems that ask you to prove that two triangles are congruent often are studied early in geometry, not only because triangles are important, but also because the *structure* of these proofs is relatively simple. In most of these problems, you can follow a straightforward plan:

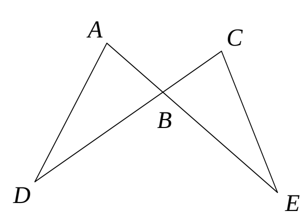
- Identify the parts of the two triangles that are known to be congruent.
- Determine whether there is enough information to prove that the triangles are congruent, and which reason (SSS, SAS, ASA, or AAS) will be used.
- Organize the information and write the proof.

Some Guided Practice

Many students feel like they're wandering around lost when they're first asked to write their own proofs. After some practice, though, finding a proof can have the feel of being a detective or working a good puzzle: one looks for clues, finds connections, and puts the pieces together. It involves the same skill that doctors and auto mechanics use in making a solid diagnosis: "Because of this, this, and this, the problem has to be *that*."

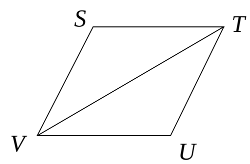
Examine each of these statements, and then prove them using one of the styles for presenting proofs described earlier.

9. **Given:** $\overline{AB} \cong \overline{BC}$ and $\overline{BD} \cong \overline{BE}$.



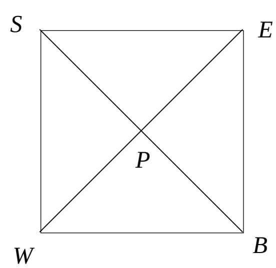
Prove: $\triangle ABD \cong \triangle CBE$.

10. Given: $\overline{SV} \cong \overline{UT}$ and $\overline{ST} \cong \overline{UV}$.



Prove: $\triangle STV \cong \triangle UVT$.

11. Given: *SEBW* is a square.



Prove: $\triangle SWB \cong \triangle EBW$.

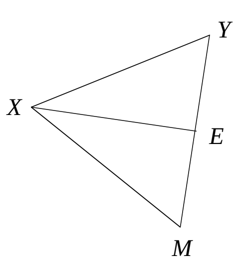
12. If one diagonal is drawn in a rectangle, then the two triangles formed are congruent.

a. Draw and label a figure for this statement.

b. Demonstrate that this statement can be proved in more than one way by writing two proofs, using different congruence postulates.

Different styles, like the four shown earlier, are not enough to make different proofs.

13. Here is a sketch of a proof. Study it; then write the whole proof in one of the styles you've seen (two-column, paragraph, or outline).



Given: $\overline{XE}$ is a median in $\triangle XMY$ and $\overline{XY} \cong \overline{XM}$.

Prove: Triangle *XEM* is congruent to triangle *XEY*.

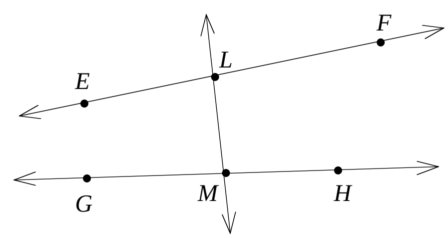
- $\overline{XE}$ is a median, so E is the midpoint of $\overline{MY}$. That means $ME = YE$.
- $XM = XY$ was given.

- $\overline{XE}$ is in both triangles.
- The triangles have three pairs of congruent sides, so they are congruent.

In an isosceles triangle, the vertex angle is the angle whose sides are the congruent sides of the triangle.

14. Draw an isosceles triangle and the bisector of the vertex angle. Prove that the two small triangles formed are congruent.

15. Here's a "proof" that shows that any pair of lines is parallel. What's wrong with it?



Suppose $\overleftrightarrow{EF}$ and $\overleftrightarrow{GH}$ are lines. Draw transversal $\overleftrightarrow{LM}$.

$\because \angle ELM$ and $\angle HML$ are alternate interior angles.

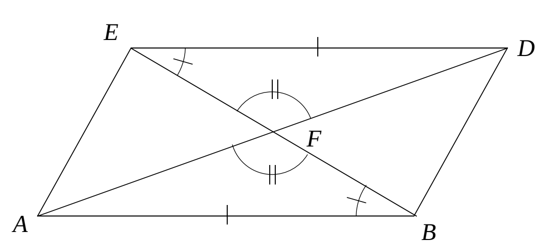
$\therefore \angle ELM \cong \angle HML$

$\therefore \overleftrightarrow{EF} \parallel \overleftrightarrow{GH}$ (alternate interior angles)

16. a. What's wrong with the following proof?

Prove:

If $\overleftrightarrow{AB} \parallel \overleftrightarrow{ED}$ and $\overline{AB} \cong \overline{ED}$, then $\triangle ABF \cong \triangle DEF$.



It's given that $\overleftrightarrow{AB} \parallel \overleftrightarrow{ED}$, so $\angle DEB \cong \angle ABE$ because parallel lines make

congruent alternate interior angles with a transversal. And $\angle AFB \cong \angle DFE$ because they are vertical angles, and vertical angles are congruent (see Problem 3 of Investigation 2.7). *And*, it's also given that $\overline{AB} \cong \overline{ED}$, so $\triangle ABF \cong \triangle DEF$ by ASA.

b. If the proof is incorrect, does that mean that the statement can't be proved? If the statement can be proved, prove it.

17. To be sure that a statement holds, mathematicians require reasoned proof. How could you convince someone that a statement like "All horses are the same color" *doesn't* hold?

What is a *prime number*?

18. Here is a famous conjecture about prime numbers. "Take any whole number, square it, add the original number, and add 41. The result is always prime."

See the examples shown in the table.

n	$n^2 + n + 41$
1	43
2	47
3	53
4	61
5	71

Is the conjecture true? Justify your answer.

Investigation 2.8

Analysis and Proof, Part 1

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On the humorous National Public Radio call-in program *Car Talk*, the following conversation took place between a caller and the hosts, "Click" and "Clack."

Caller: Yesterday the van did the same thing. As soon as we got above 45 miles per hour, it started shuddering again, only worse. Immediately and more dramatically.

Clack: I could've guessed that . . . You said you feel the vibration in the steering wheel?

Caller: Yes, very much in the steering wheel.

Click: Good, good, good. What color is this van? Is it one of the earth tones?

How does a doctor figure out what's causing your cough? How does a mechanic figure out why your car won't start? How does a chess player decide on the next move? How does a mathematician find a proof for a conjecture? These are not easy questions to answer. In this lesson you'll learn some techniques for coming up with mathematical proofs of geometric facts.

1. **Write and Reflect** Think about a time when you had to diagnose a situation and find its underlying causes. Write about what you did and how you looked for clues.

Understanding or Writing: Which Comes First?

Another real question is: "How do you find a statement to prove?" Getting good at this is another important part of doing mathematics. It comes with practice.

You've had some practice *writing* proofs. The *real* question is:

> *How do you* think up *a proof in the first place?*

Creating a proof is sometimes called the "analysis of the proof." Analysis is always necessary before writing a proof, but sometimes it is very brief and goes on in your mind, almost without your noticing it. If you see the logic underlying the proof, then writing the proof is just a matter of expressing yourself clearly. But what if you *don't* yet see the connections? What if you see a lot of facts and clues, but they don't seem to point to a solution? How, then, do you begin?

This section of the module offers you three techniques of analysis for proofs:

Ideas like these can help you analyze other complex situations as well.

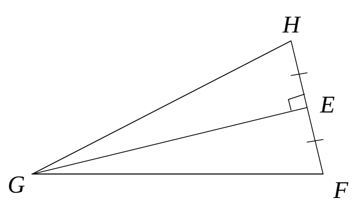
- the visual scan,
- the flow chart, and
- the reverse list (You'll see this in the next investigation.)

You may find that one method makes the most sense to you and becomes your main tool for analysis. To be skilled at analysis, however, you will need to use all three techniques. In fact, if you are having trouble developing a proof using one method, it can be quite helpful to switch from one technique to another.

THE VISUAL SCAN

A visual scan is a careful examination of the model (usually a drawing) created for the problem. After a sketch is drawn, mark it to show all known congruent parts and any other known information. Mark additional parts that you conclude are congruent. In studying the picture, a strategy for the proof may become clear to you.

For example, given that E is the midpoint of $\overline{HF}$ in $\triangle FGH$, and that $\overline{EG}$ is perpendicular to $\overline{HF}$, prove that $\overline{HG} \cong \overline{FG}$.



2. First sketch a figure, showing the given information visually. Studying the marked figure suggests a strategy for this proof:

- Show that $\triangle HEG \cong \triangle FEG$ by SAS.
- Then conclude that $\overline{HG}$ and $\overline{FG}$ must be congruent because they are corresponding parts of congruent triangles.

Write the full text of this proof as outlined.

CORRESPONDING PARTS OF CONGRUENT TRIANGLES

This strategy is simple and direct. Does it work only for triangles?

In the visual-scan example, the two segments were proved congruent by showing that they are corresponding parts of congruent triangles. The CPCTC strategy is used in so many proofs that many textbooks devote an entire section to it. In order to show that two segments (or angles) are congruent:

- First find two triangles that contain the segments, and prove the triangles congruent.
- Then conclude that the segments are congruent because they are corresponding parts of congruent triangles.

Keep this strategy in mind as you work through the rest of this section, and see how often it appears as you continue your study of congruence.

3. Prove that the base angles of an isosceles triangle are congruent.

- First sketch a figure.
- Draw the median from the vertex angle to the base, and then use the SAS postulate to show the two new triangles formed are congruent.
- Use CPCTC to show that the base angles of the original triangle are congruent.

Write the full text of this proof as outlined.

FOR DISCUSSION

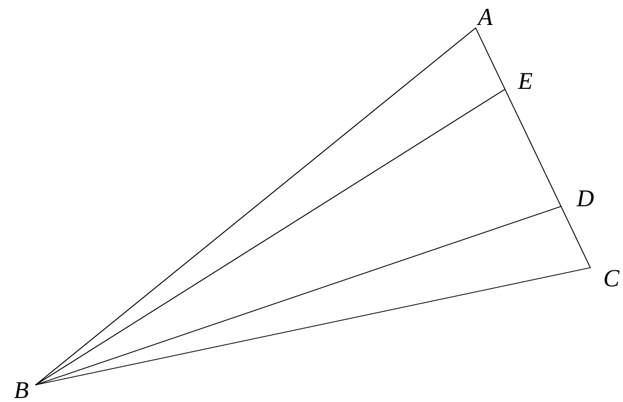
The CPCTC statement serves a somewhat different purpose in proofs from statements like SSS or SAS. Discuss how each kind of statement is useful in proofs and how their uses differ.

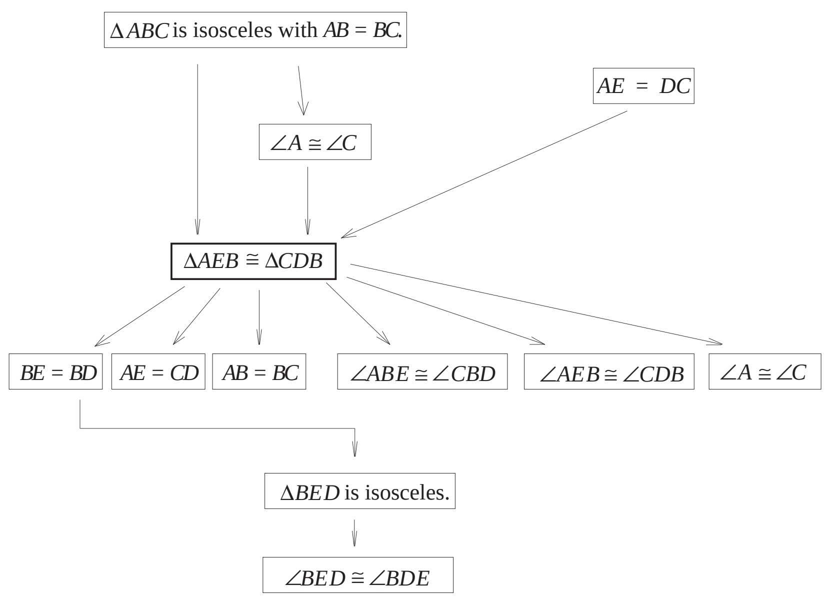
THE FLOW CHART

The flow chart is a "top-down" analysis technique. At the top of the chart, write what you know is true. Below each statement, write conclusions based on that statement. Keep working down until the desired conclusion is reached. For example:

Given: Isosceles $\triangle ABC$ with $AB = BC$ and $AE = CD$.

Prove: $\angle BDE \cong \angle BED$.





In referring to the flow chart as you write the proof, leave out unnecessary steps and information. A readable proof will take the most direct route through the flow chart.

4. Write the proof that is described by this flow chart.

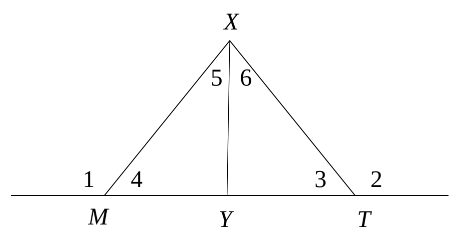
A flow chart has several advantages. It gives you a way to start investigating and organizing what you know, even if the entire proof still is unclear to you. When it is done, it forms an outline for the steps of a written proof. Also, extra information that

you might write down may be helpful in generating ideas for alternate ways to write the proof.

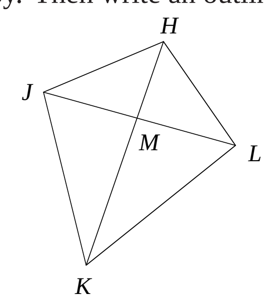
5. Make a flow chart for a proof of the following statement:

Given: $\angle 1 \cong \angle 2$, $\overline{XY}$ bisects $\angle MXT$.

Prove: $\overline{MY} \cong \overline{TY}$.



6. Use a visual scan to analyze this proof. Copy the figure and mark the given information on your copy. Then write an outline for the proof.



Given: $\overline{HJ} \cong \overline{HL}$ and $\overline{JK} \cong \overline{LK}$.

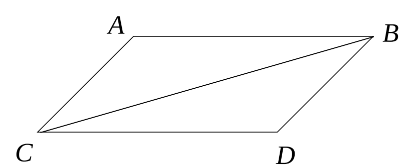
Prove: $\triangle HJM \cong \triangle HLM$.

To prove *ABCD* is a parallelogram, you will need to show that $\overline{AD} \parallel \overline{CD}$ so that you will have both pairs of opposite sides parallel. A diagonal is drawn because it might help you.

7. Use whichever method of analysis you prefer to plan the following proof. Then write a complete proof.

Given: $\overline{AB} \cong \overline{CD}$ and $\overline{AB} \parallel \overline{CD}$.

Prove: $ABCD$ is a parallelogram.



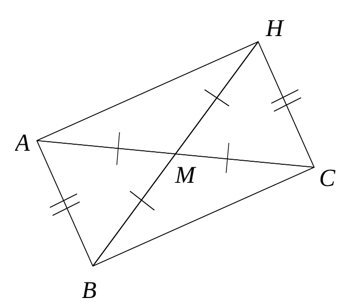
You have proved that if a pair of opposite sides of a quadrilateral are both congruent and parallel, the quadrilateral is a parallelogram. Look for places where you can use this result in future proofs.

8. Below is a figure that someone marked for a "visual scan," but an error was made.

a. Find the error in the marking.

b. What incorrect conclusion in a proof might result from this error in marking the figure?

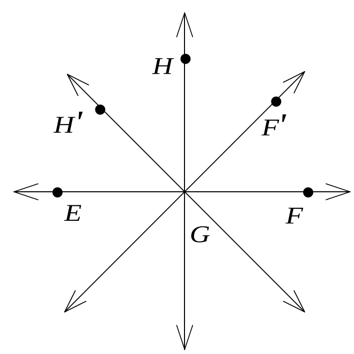
Given that $\overline{AC}$ and $\overline{BH}$ bisect each other and $\overline{AB} \cong \overline{CH}$, what type of quadrilateral is $ABCH$?



CHECKPOINT

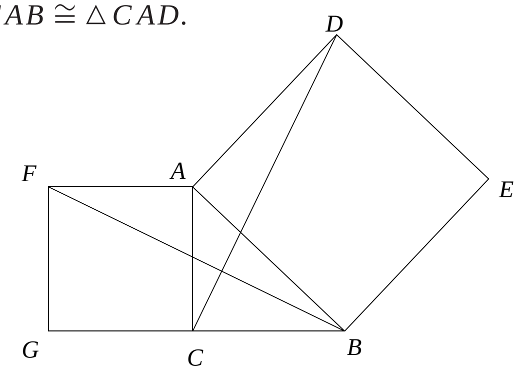
9. **Given:** $\overleftrightarrow{GF} \perp \overleftrightarrow{GH}$ and $\overleftrightarrow{GF'} \perp \overleftrightarrow{GH'}$.

Prove: $\angle F'GF \cong \angle H'GH$.

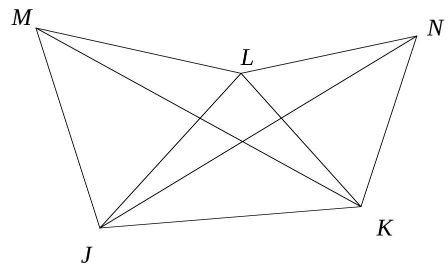


10. **Given:** *FACG* and *DABE* are squares.

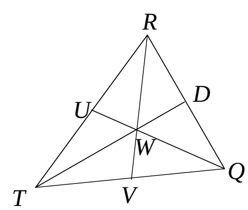
Prove: $\triangle FAB \cong \triangle CAD$.



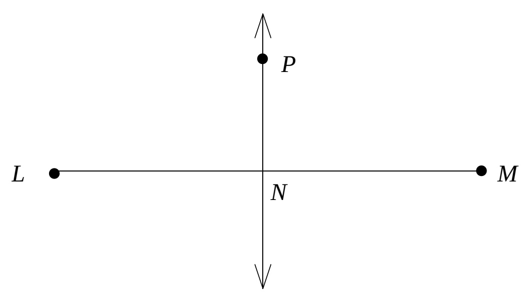
11. Given that triangles $\triangle LJM$ and $\triangle LNK$ are equilateral, prove that $\overline{MK} \cong \overline{JN}$.



12. In isosceles triangle QRT, $QT = TR$. $\overline{VR}$ and $\overline{UQ}$ are medians. Prove that they have the same length.

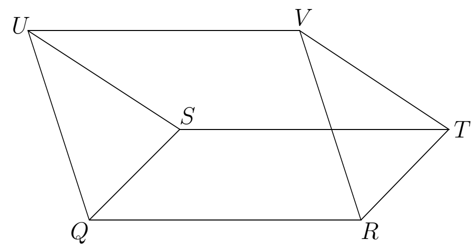


13. P is a point on the perpendicular bisector of $\overline{LM}$. Prove that $PL = PM$.



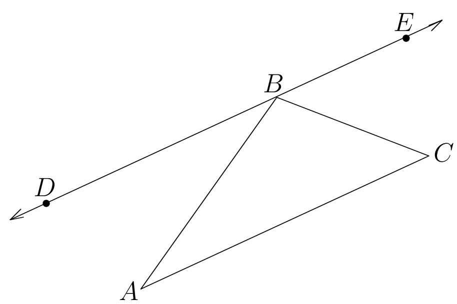
TAKE IT FURTHER. . . . .

14. In this figure, $QSTR$ and $STVU$ are parallelograms. Prove that $\triangle QSU \cong \triangle RTV$.



15. In the plane, it is possible to construct a parallel to any side of any triangle (as in the figure below) in which $\overleftrightarrow{DE} \parallel \overline{AC}$. Use this construction to write a proof that

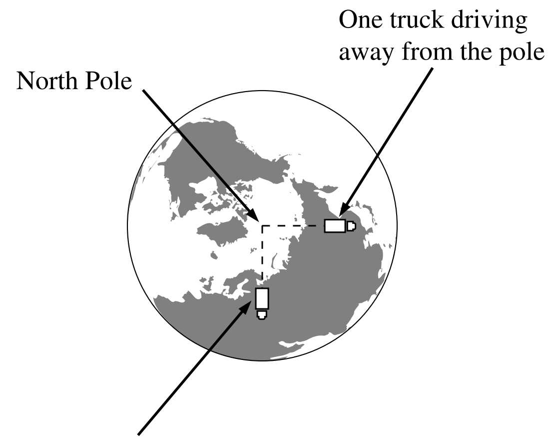
$$m\angle A + m\angle C + m\angle ABC = 180^\circ.$$



You've just shown that all triangles have angle measures that sum to 180°. So you would expect that any triangle you drew, no matter how big, would have angle measures that sum to 180°. For example, surely a triangle drawn accurately on your school parking lot . . .

16. But what if your parking lot were *very* big? Suppose, in fact, that your school paved all of the northern hemisphere. On this enormous parking lot, two trucks at the North Pole start at right angles to each other and drive due south to the equator, painting lines as they go.

A third line is painted along the equator to connect the two free ends, completing a triangle. What is the sum of the measures of the angles in *this* triangle?



17. Explain what you found in Problem 16. How can it contradict your *proof* that the sum of the measures of the angles in a triangle is 180°? What's going on?

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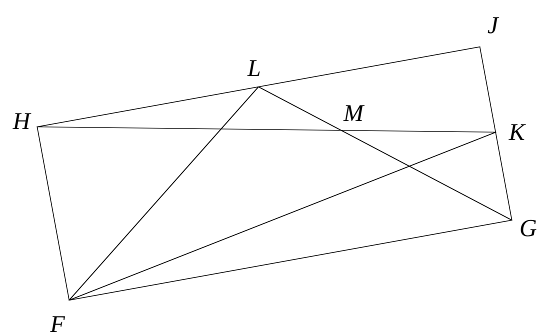
18. **Challenge** $FGJH$ is a rectangle and L and K are midpoints. Prove that $\angle LMH \cong \angle LFK$.



Investigation 2.9

Analysis and Proof, Part 2

Hypothesis and Conclusion

Usually, statements don't come your way in the form

Given: Isosceles triangle ABC with $AB = BC$.

Prove: $\angle A \cong \angle C$.

Instead, you are trying to be certain about things that you suspect are true (maybe from the results of an experiment), and you express yourself in something much closer to English. You say things like "Vertical angles are congruent" or "Base angles of an isosceles triangle are congruent" or "If it rains this afternoon, then there won't be soccer practice."

The hypothesis is what you are assuming. The conclusion is what you're trying to *conclude* from the hypothesis.

When you see a statement written like these, how do you know what to prove? One way is to break the sentence into a *hypothesis* and a *conclusion*. The sentence "Vertical angles are congruent" can be rewritten as "If two angles are vertical angles, then they are congruent." The hypothesis is "two angles are vertical angles" and the conclusion is "the angles are congruent."

Ways to Think About It

Here are two rules of thumb for recognizing each part of a statement of this type:

Often the word "then" is understood and not stated, as in "If it rains this afternoon, there won't be soccer practice."

- If the sentence is in "if-then" form, the "if" clause is the hypothesis and the "then" clause is the conclusion.
- If the sentence isn't in if-then form, the hypothesis is formed from the subject and the conclusion is formed from the predicate (generally at the "conclusion" of the sentence).

Examples:

Sentence	Hypothesis	Conclusion
If two parallel lines are cut by a transversal, the alternate interior angles are congruent.	two parallel lines are cut by a transversal	the alternate interior angles are congruent
The base angles of an isosceles triangle are congruent.	base angles of an isosceles triangle	are congruent
Two triangles with the same area are congruent.	two triangles with the same area	are congruent
Congruent triangles have the same area.	congruent triangles	have the same area
People with large hands have large feet.	people with large hands	have large feet
Out of sight, out of mind.	something is out of sight	it is also out of mind

1. Of course, the fact that a sentence states a conclusion doesn't mean that the conclusion is *true*. Which statements in the table are not necessarily true?

2. In each sentence, identify the hypothesis and conclusion.

a. If two lines make congruent alternate interior angles with a transversal, then the lines are parallel.

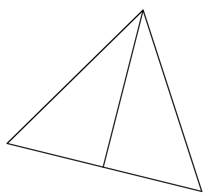
b. If n is any whole number, $n^2 + n + 41$ is a prime.

c. Two triangles are congruent if two sides and an included angle of one are congruent to two sides and an included angle of the other.

d. Two lines parallel to a third line are parallel to each other.

e. The area of a rectangle is the product of its length and its width.

WHAT AM I TRYING TO PROVE?



If you are given a statement to prove, then the hypothesis of the statement is what is "given," and the conclusion of the statement is what you have to prove. For example, suppose you want to prove the statement "the base angles of an isosceles triangle are congruent." The given would be that you *have* an isosceles triangle (and you could draw a picture of that). The thing you would want to prove is that its base angles are congruent.

For each of the following statements, draw a picture illustrating the hypothesis (the given); then decide whether the statement is true. For those that are not true, give a counterexample. For those that are true, give a proof.

3. Two lines that are perpendicular to the same line are parallel to each other.

4. If a line bisects an angle of a triangle, then it bisects the opposite side.

5. Any two medians of an equilateral triangle are congruent.

6. Equilateral triangles are equiangular.

7. Equilateral quadrilaterals are equiangular.

8. Equiangular triangles are equilateral.

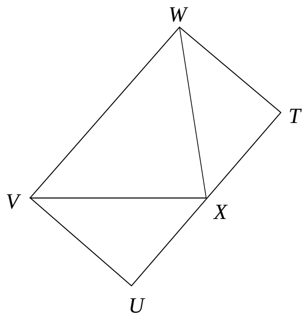
9. Equiangular quadrilaterals are equilateral.

10. If a triangle has two congruent angles, it is isosceles.

THE REVERSE LIST

The reverse list is "bottom up" rather than "top down."

In the visual scan and the flow chart, you start with what you know and work toward the final conclusion you are trying to prove. The *reverse list* works in the opposite direction. Start at the end and work backwards, repeatedly asking "What do I NEED?" and then "What can I USE to prove that?"



How do I choose *which* sides for this "need"? A careful sketch often helps.

How do I choose which triangles for this need?

Before getting more precise, study this example:

Given: *TUVW* is a rectangle; *X* is the midpoint of $\overline{TU}$.

Prove: Triangle *XWV* is isosceles.

To prove that triangle *XWV* is isosceles:

- NEED: triangle *XWV* is isosceles.
- USE: A triangle is isosceles if two sides are congruent.
- NEED: $\overline{XW} \cong \overline{XV}$
- USE: CPCTC
  - NEED: congruent triangles; choose $\triangle WXT \cong \triangle VXU$.
  - USE: SAS
    - ∗ NEED first side: $\overline{TW} \cong \overline{UV}$
    - ∗ USE: Opposite sides of rectangle are congruent.
      - · NEED: *TUVW* is a rectangle.
      - · USE: Given
    - ∗ NEED angle: $\angle T \cong \angle U$
    - ∗ USE: All angles of a rectangle are right angles, so they are congruent.
      - · NEED: *TUVW* is a rectangle.
      - · USE: Given
    - ∗ NEED second side: $\overline{TX} \cong \overline{UX}$
    - ∗ USE: Midpoint divides segment into congruent parts.
      - · NEED: *X* is the midpoint of $\overline{TU}$.
      - · USE: Given

11. A complete analysis outlines the proof for you in reverse order. Write the proof for the example given above.

How do you decide what statements to put in the USE spots? There is no foolproof method, but there *is* a straightforward way to narrow down the number of statements to try.

What you can do is look in your notes, the Student Module, and your memory, and find all the previously-established results that have what you NEED as a conclusion. For example, one of the NEED statements in the above analysis is $\overline{TX} \cong \overline{UX}$. You should say to yourself:

> "I'm looking for a previously-established result that has 'congruent segments' as a conclusion."

Here are a few:

- An isosceles triangle has two congruent sides.
- Corresponding parts (sides) of congruent triangles are congruent.
- The midpoint of a segment divides it into two congruent segments.

Now that you have the narrowed list, you can take each statement, one by one, and decide if you really can use it. How? Take its *hypothesis* and turn it into a NEED, and see if you can establish *that*. In this case, you'd go down the list and ask:

a. Can I get $\overline{TX}$ and $\overline{UX}$ as sides of an isosceles triangle?
- No; they aren't even sides of the same triangle.

b. Can I get $\overline{TX}$ and $\overline{UX}$ as corresponding sides of congruent triangles?
- The only ones that would work are $\triangle XTW$ and $\triangle XUV$, but the reason I want $\overline{TX} \cong \overline{UX}$ in the first place is so that I can *prove* these triangles congruent.

c. Can I get $\overline{TX}$ and $\overline{UX}$ as pieces of a segment that is divided by a midpoint?
- Yes; I'm given that X is the midpoint of $\overline{TU}$.

When you're doing this kind of analysis, here are some points to remember:

- "Given" information can be used anywhere you want as a USE.
- "CPCTC" can be a USE for both congruent segments and congruent angles.
- There are only four ways to get congruent triangles right now: SSS, SAS, ASA, and AAS.
- This method almost always works, but before you hit on the right path, you may go down some dead ends.

Prove each of the following statements.

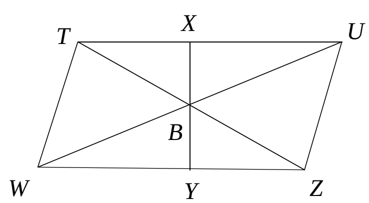
12. Opposite angles of a parallelogram are congruent.

13. Opposite sides of a parallelogram are congruent.

14. The diagonals of a parallelogram bisect each other.

Analyzing proofs can be like driving without a roadmap. It is easy to take wrong turns, or find that the path you're on doesn't go anywhere.

15. Here is a "reverse list" analysis that has led to a dead end. Study it, and then finish the analysis by choosing a different path that works better.



Given: $TUZW$ is a parallelogram.
Prove: $\triangle TXB \cong \triangle ZYB$.

NEED: $\triangle TXB \cong \triangle ZYB$

USE: SSS

- NEED: side $\overline{TB} \cong \overline{ZB}$
- USE: diagonals of a parallelogram bisect each other.
- NEED: side $\overline{TX} \cong \overline{ZY}$
- USE: ???

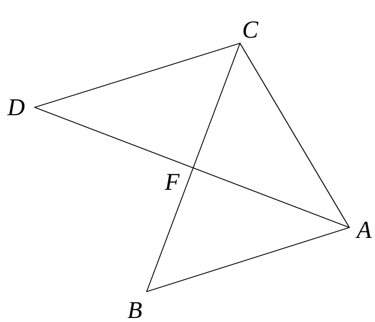
Dead end!

16. In the figure on the left, $\overline{AD}$ is the perpendicular bisector of $\overline{BC}$. With this given information, one pair of triangles can be proven congruent.

a. Name the pair.

b. Prove that they are congruent.

c. Describe how you analyzed the proof.



17. **Challenge** If two medians of a triangle are congruent, then the triangle is isosceles.

18. **Challenge** If two angle bisectors of a triangle are congruent, then the triangle is isosceles.

19. **Write and Reflect** Which of the three analysis techniques (visual scan, flow chart, or reverse list) do you like the best? Like the least? Explain why.

20. Explain how you usually analyze problems. Which of the three techniques most closely resembles your own way of thinking? In answering, consider how you might analyze a fairly complex problem in your own life—like making a tough decision.

21. Flow charts are used in computer programming, debates, and many other fields as a way to organize information. Find an example of a flow chart and write an explanation of its purpose and contents.

Perspectives on Proof

Reading from Leila Schneps

In these essays, two contemporary mathematicians write about proof and their experiences learning mathematics. How do their approaches to proof differ?

Leila Schneps is a mathematician currently living in France. Here she shares some of her thoughts about mathematics and how she came to the research she is now doing.

Mathematicians often describe mathematics like a landscape, half-hidden under a swath of mysterious cloud. The beauty and attraction of mathematics is the desire to wander about that landscape until it becomes as familiar to you as your own garden; as you explore, you get more and more of a feel for its different features. The most difficult aspect of mathematics is becoming sufficiently familiar with the terrain you're exploring to be able to locate the problems in it, the holes and gaps in your understanding which call out to be filled.

Learning to find your way around in mathematics is like visiting a foreign city with no map; it is much harder than actually guessing the answers to specific problems and then proving that they are right once you have them. That is why it's essential to learn mathematics with teachers who treat it as a landscape to explore, not just as a series of problems. Your teacher can serve as your guide until you learn how to find your own way about.

Doing mathematics is essentially trying to make out the nature, the properties, and the interrelations of abstract (or sometimes, as in geometry, even relatively concrete) objects. When trying to figure out how they work, you can start by making conjectures, to see if you find something that stands up to experiment. Up to a certain point, you can continue to leap ahead from conjecture to conjecture in sketching out your vision. But there will be a point when lack of proof just comes down to a lack of understanding *why* the things you are guessing actually are true, and that will prevent you from going any farther.

A mathematical experiment is often a calculation.

When you are faced with a specific conjecture—something you feel is right for intuitive or experimental reasons—and you need to prove it in order to deepen your understanding and pursue it further, there are essentially two ways you can go about the proof.

You can start by using every technique that looks even remotely related to the problem: translate into other terms, use analogy, make estimations, transform the problem into something that reminds you of something else, and then try the method used to prove that something else. After a series of such transformations, you may end up with a proof of what you wanted, but most often, anyone who reads it (including you yourself)

may feel that after following every logical step, they still don't really understand *why* the statement is true.

The other way is to continue to stare at your whole situation and context, trying to get to know it better. You may not even try immediately to obtain a proof of any one statement. You start rather far away from the difficult point, and you make sure you are building your house on firm ground, that the definitions and basic properties are all there, all solidly leaning against each other. You just continue to feel the objects, turning them over and over in your fingers as it were, finding out their new properties, their analogies, their relations to each other. You realize that different phenomena you noticed are actually natural consequences of the same thing. Your picture of the "landscape" slowly grows clearer and clearer as the fog seems to lift. One day, you see it just as clearly as if you were looking out the window. Everything is flooded with sunshine and you simply can't imagine what it was that you didn't understand before. This was the kind of proof favored by Professor Alexander Grothendieck; he called it "proof by the rising sea."

The difficulty with this second method of proof is writing it down. The language of mathematics is not completely adequate to express this vision in its entirety. You meet with the same problems as if you were trying to describe a painting to someone who hadn't seen it: you can say a lot, in many words, but one look at the painting (if it were only possible) would make all those words needless. Mathematical writing is necessarily a linear, step-by-step procedure, even when the real need is to convey the feeling that "it all works together to create a single idea." Therefore, each time you read of someone else's vision, you have to recreate it in your own way, in your own mind . . . and the process of research begins again.

When I was in high school, I neither liked nor disliked mathematics. I liked solving the problems—sometimes. I was quite pleased when my answer turned out to be different from the one in the book. But nobody ever gave me the slightest idea that mathematics was anything more than a lot of techniques that might be useful for becoming an engineer, or for doing physics to find out about the real world, say.

I went to university never dreaming of majoring in mathematics, but during my first semester, everything changed. I took mathematics, physics, literature, and languages because I thought I should try some of everything before choosing a single area of study. My very first professor of mathematics was Andrew Wiles. He taught calculus, and although the curriculum was basically the same as in high school, the course did not resemble anything I had ever seen. I don't think he needed to do anything particular to achieve that effect; he explained things the way he saw them and it was real mathematics. So, I decided during that first semester to study mathematics, and

My feeling that I was hearing a great mathematician was justified—14 years later, Wiles proved Fermat's Last Theorem!

stuck to it through thick and thin for the next eight years. And there was plenty of thick—for months at a time I got lost in new domains without the slightest idea of the way out—or around.

At that point a friend, casually encountered in the street, gave me a typewritten manuscript by Alexander Grothendieck, which I read and fell in love with. I wanted to contact Grothendieck, but he had disappeared shortly before; that discovery was the beginning of my effort to seek out, decipher, and continue to explore the ideas in other unknown manuscripts of his. My efforts have taken me to different cities, towns, and precariously-perched mountain villages all over France.

At the time of this writing, Grothendieck's whereabouts are still unknown.

Reading from Hung-Hsi Wu

Hung-Hsi Wu is a mathematician who teaches and does research at the University of California at Berkeley. In addition to his mathematics research, Professor Wu is extremely interested in mathematics education. Here are his comments on the importance of proof in mathematics.

Proofs in mathematics serve the basic function of verifying that something is true. In most cases, they also *explain* why it is true.

Mathematicians are obsessed with the need to know whether something is true. In mathematics, "true" means true *always* and *without any exceptions*. Contrast this with the fact that almost all "true" statements in everyday life do have exceptions.

The mathematician's desire for truth in such an absolute sense, with its insistence on proof at each step, may be puzzling to outsiders. However, its roots are deeply embedded in ordinary human experiences. As we go through life, we are constantly reminded of the fact that things are not always what they seem. As children, we watch the sun move across the sky every day and see no reason to doubt our senses, but we later learn that, in fact, it is the earth that moves around the sun. We also grow up believing in the solidity of the earth under our feet, only to be shocked by the discovery later that the solid earth is quite capable of the most violent tremors.

We also learn *why* we developed the illusion that the sun revolved around the earth: it is caused by the earth's own rotation around its axis.

Even something so simple as the drinking of a glass of water could be a cause for some soul searching. Where I grew up in China, the drinkability of a glass of seemingly clean and pure water depended on whether the water had been boiled. Could I tell the difference between boiled and unboiled water? Not at all. Yet I was to learn about

the lethal bacteria that could reside in the purest looking water, and so I learned to distrust appearance alone.

Perhaps experiences of this type help shape our imaginations and drive us all to dig beneath the surface in search of the truth. But whatever the reasons, the thirst for truth is real and universal. It is just that in the clamor for proofs in mathematics, one witnesses this thirst in its purest form.

Proofs bring order out of the seeming chaos that hangs over the thousands and thousands of mathematical theorems, revealing the interrelationships among them. These interrelationships give the mathematical ideas cohesion and structure, and I believe it is only through proofs that these interrelationships can be perceived and understood.

Fermat's Last Theorem is easy to state and understand: for any integer $n > 2$, there are no positive integers x, y, and z so that $x^n + y^n = z^n$. It was not so easy to prove! The recent proof by Andrew Wiles is the culmination of more than three centuries of mathematical developments and uses the most advanced techniques in modern mathematics. The work of Fermat and Wiles is discussed in the *Connected Geometry* module *Optimization*.

The *incircle* is the circle inside the triangle tangent to all three sides. The *nine-point circle* is the circle that passes through the midpoints of all three sides. The reason for the name is that this circle also passes through six additional points of special interest.

Therefore, it makes no more sense to me to discuss or teach higher mathematics outside the context of proofs than to try to sample English literature through the use of a dictionary. For example, a person with no understanding of proof in mathematics might mistake a statement such as Fermat's Last Theorem to be simpler than the much more complicated-looking quadratic formula. For this reason, any real understanding of mathematics requires a thorough knowledge of its culture of proofs.

The most satisfying aspect of proof is that it shows *why* something is true, no matter how surprising it may seem. A good example is the theorem in geometry called Feuerbach's theorem, which says that the "nine-point circle" and the "incircle" of a triangle are always tangent to each other. This is not a theorem one would readily believe without a thorough examination of its proof. Even with good geometry software, I doubt that one would be willing to concede its truth just by repeated experimentations on the computer screen.

Examples at least as striking as Feuerbach's theorem abound in the more advanced portion of mathematics. Because of the abstract nature of some of these theorems—try, for example, to imagine the geometry of a space of not three but *infinite* dimensions!—or because of their complexity, it is difficult at times to verify even their special cases on a computer. For this reason, one can rely only on proofs for conviction. However, I must not overstate my case: most proofs are intuitively persuasive (once you understand them), but not all are. In the same breath, it would be fair to state that almost every mathematician has the unshakable belief that, in those cases where persuasive proofs do not as yet exist, they will when real understanding is attained.

I began this essay by discussing the universal character of the need for proofs. It would only be appropriate therefore if I conclude by mentioning one possible effect that the study of proofs may have outside of mathematics. A mathematical proof, when

correctly executed, is totally objective: personal preferences or emotional biases play no role in its outcome. If it is correct, a proof gives the reader an undeniable conclusion, no matter how fantastic or unbelievable.

Learning about proofs therefore fosters a predisposition to accept the edicts of reason. From this, it is but a small step to learn to accept facts without prejudice, and therewith, to lead a life without delusions. It would be farfetched to trivialize such an arduous journey—the attainment of a way of life without delusions—by claiming that the learning of proofs is the correct first step. All the same, the learning of proofs can provide good training in the right direction. Somerset Maugham once wrote a gentle satire of a woman who does not "know the truth when she sees it." Anyone well versed in the art of proof would certainly know the truth when he or she sees it.

See Maugham's short story "*Jane*."

Investigation 2.11

Investigations and Demonstrations

Here you will find eight investigations designed to help you refine your understanding of triangles, congruence, and proof. Here are some general guidelines for you to follow as you work on any of the investigations:

- **Explore** the problem. Use hand or computer drawings to help you understand what the problem statement means.
- **Explain** what you observe. If you can, justify what you say with a proof. If you can't find a complete proof, describe what you did figure out, and say what's missing from your proof.
- **Summarize** your work. Include drawings, conjectures that you made, a list of important vocabulary words, theorems, rules, or ideas that came up in the investigation, and questions that require further exploration.

Perpendicular Bisectors

The Investigation

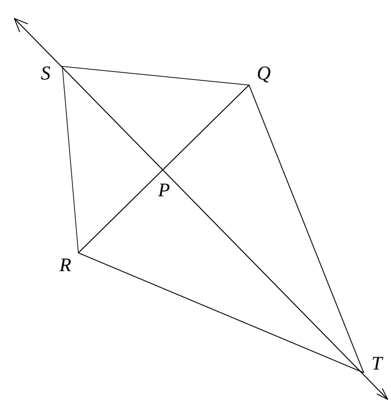
Verify and prove this theorem: "Any point on the perpendicular bisector of a line segment is equidistant from the endpoints of the line segment."

Take It Further. . . . .

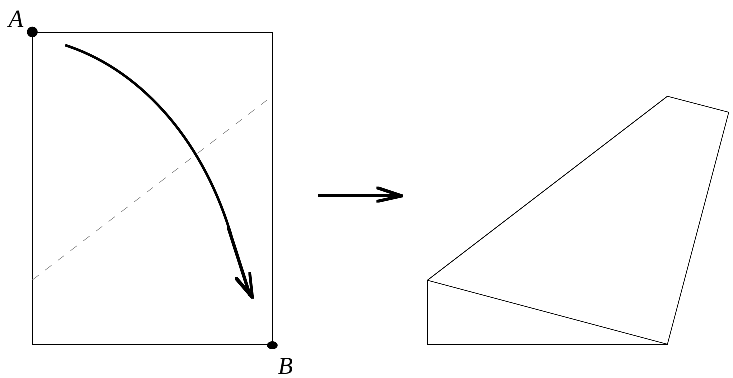
To get the converse, switch the "if" (hypothesis) and "then" (conclusion) parts of the theorem: "*If* a point is equidistant . . . *then* the point is on the . . . ."

1. You've proved the following theorem: "If a point is on the perpendicular bisector of a segment, then the point is equidistant from the segment's endpoints." Can you prove the theorem's converse?

2. In this figure, $\overleftrightarrow{ST}$ is the perpendicular bisector of $\overline{RQ}$. Prove that $\angle SRT \cong \angle SQT$.

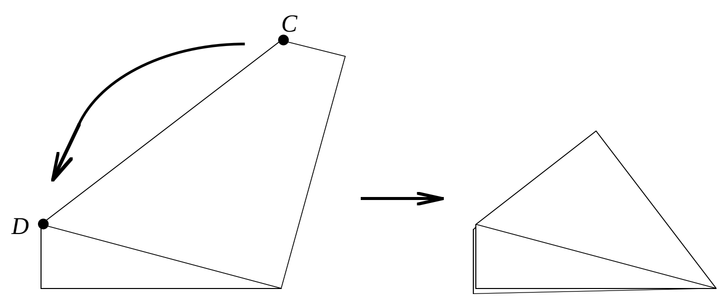


3. This paper folding construction starts with a rectangular sheet of paper.



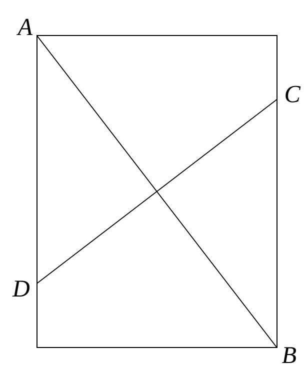
Fold A onto B and crease.

Then . . .



. . . fold C onto D and crease.

Unfold the paper. The creases should look like this:



Prove that any point on $\overline{CD}$ is the same distance from A as from B.

It may help to construct examples of triangles in which the perpendicular bisector of at least one side does pass through the opposite vertex, and other triangles in which it does not.

4. Describe the set of triangles which have:

a. one side (but no other) whose perpendicular bisector passes through the opposite vertex;

b. two (but not three) sides whose perpendicular bisectors pass through the opposite vertex;

c. three sides whose perpendicular bisectors pass through the opposite vertex.

Explain what you find.

5. Using a cup or glass, trace a circle on paper. Explain how to find the center of the circle.

What does *algorithm* mean?

6. Give an algorithm for constructing a circle that will pass through the vertices of a given triangle.

If you use geometry software, you can make one construction and just drag one vertex around.

7. Draw several triangles. In each, construct the perpendicular bisectors of all three sides. Notice that all three perpendicular bisectors meet at a point. Does this always happen? Provide a proof or a counterexample.

8. A circular saw blade fell and broke. All you can find of it is a piece that looks like this. Explain how to find the diameter of the blade so you can buy a new one.

ANGLES AND SIDES IN TRIANGLES

THE INVESTIGATION

The sizes of the angles in a triangle are related to the relative sizes of the sides. For example, if a triangle has two congruent sides, it must also have two congruent angles.

You may have proved this statement in Problem 3 of Investigation 2.8. If so, look up your old proof. Some people use the abbreviation BAITC for the statement you're proving. What might "BAITC" stand for?

- Prove this fact.
- "Strengthen" the conclusion. (You can specify *which* angles must be congruent.)
- What if a triangle is scalene? Could it have two congruent angles? Develop a conjecture about how to locate the largest, smallest, and "middle-sized" angles in a triangle. How could you prove your conjecture?

TAKE IT FURTHER. . . . .

9. From the SAS postulate, we know that if two sides and an included angle of one triangle are congruent to two sides and the included angle of another triangle, then the third sides are also congruent.

Suppose, instead, that two sides of one triangle are congruent to two sides of another, but the included angle in the first triangle is larger than the included angle in the second. Then what can you conclude about the third sides?

Why is this result sometimes called the "Hinge Theorem?"

10. Show that if two *right* triangles have the hypotenuse and a leg of one congruent to the hypotenuse and a leg of the other, the triangles are congruent.

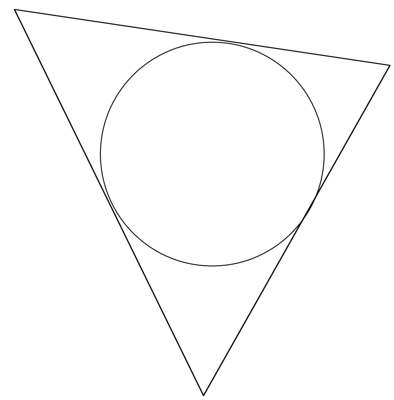
This test is sometimes called "Hypotenuse-Leg" and is abbreviated "HL."

11. Carefully locate several points that are equidistant from the *sides* of an angle. What would you get if you were able to locate *all* the points equidistant from the sides of an angle? Prove what you say.

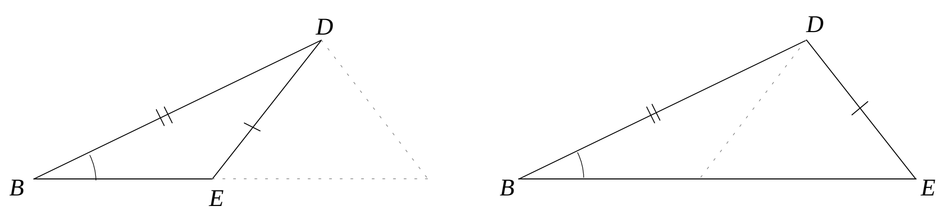
Use computer, pencil, paper folding . . .

A circle is *inscribed* in a triangle if it is tangent to all three of the triangle's sides.

12. Draw a triangle. Figure out how to *inscribe* a circle in the triangle using compass and straightedge or geometry software. Write an explanation of how you did it.



SSA There is no "SSA" triangle postulate, because two sides and a non-included angle of one triangle can be congruent to two sides and a non-included angle of another triangle *without* the two triangles being congruent.



But . . .

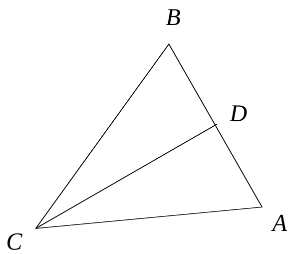
13. Could there be an "SSA" Postulate? What if you knew not just the non-included angle's measure but also that it was the *largest* angle in each triangle? That is, what if two sides and the largest non-included angle of one triangle were congruent to the corresponding two sides and largest angle of the other? Then, can you conclude that the triangles are congruent?

If this is so, this would be a generalization of HL. Why?

14. And what about an "SSa" Postulate? That is, what if you also knew that the non-included angle was the *smallest* angle in the two triangles? Could you then conclude that the triangles are congruent?

ISOSCELES TRIANGLE PROOFS

THE INVESTIGATION



Here are four statements:

1. Triangle ABC is isosceles (base $\overline{AB}$).
2. $\overline{CD}$ is a median.
3. $\overline{CD}$ is an altitude.
4. $\overline{CD}$ bisects $\angle ACB$.

Show that, if any two of the statements are given, the other two statements can be proved. For example,

Given: statements 1 and 2,

Prove: statements 3 and 4.

Continue writing statements (and proofs) until you have verified that any two will prove the remaining two. How many theorems do you have?

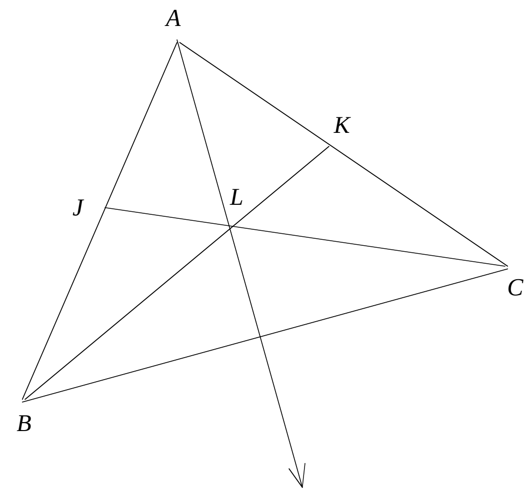
Write up your work for the investigation. Organize your sketches, notes, questions, ideas, and proofs.

TAKE IT FURTHER. . . . .

15. The investigation lists four statements. Invent *one* statement (about $\triangle ABC$ or about triangles *ADC* and *BDC*) that guarantees the four statements in the investigation are true.

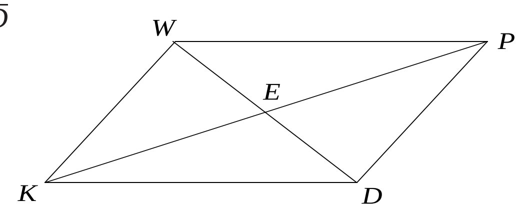
The congruent sides of an isosceles triangle are called its *legs*.

16. **Challenge** $\triangle ABC$ is isosceles (base $\overline{BC}$), and J and K are midpoints of the congruent sides. If $\overline{JC}$ and $\overline{KB}$ intersect at L, show that $\overrightarrow{AL}$ bisects $\overline{BC}$. Does the same thing happen if the triangle is not isosceles?



17. Here are six statements about quadrilateral $WPDK$.

- $WP = KD$
- $WK = PD$
- $ED = EW$
- $KE = EP$
- $\overline{WP} \parallel \overline{KD}$
- $\overline{WK} \parallel \overline{PD}$



a. Find two statements that give sufficient information about $WPDK$ to allow you to prove the rest.

b. Are *any* two sufficient to prove the rest?

c. Invent a seventh statement about $WPDK$ that would make the other six true.

A Reflection Puzzle

The Investigation

Develop a conjecture about how many reflections are needed. Test it well. When you are convinced that you have found the minimum number, use paper triangles or a computer drawing tool to demonstrate that your answer is correct.

Draw two congruent triangles, anywhere in the plane. By using only reflections, take Triangle 1 and "map" it to Triangle 2. That is, apply reflections (and only reflections) to one triangle to superimpose it on the other triangle. Try other examples until you have a solid understanding of how the image of a triangle "moves" when you apply a succession of reflections to it.

If two congruent triangles are drawn anywhere in the plane, what is the minimum number of reflections required to map one onto the other?

Take It Further

18. Write a set of directions that tell which reflections to use to achieve the minimum number of reflections that you discovered in the above investigation.

19. How many reflections does it take to map one equilateral triangle onto another triangle, congruent to the first and placed anywhere in the plane? Will this require fewer steps than the method you discovered in the above investigation?

20. Find several different ways to draw two triangles that require only *one* reflection to map one onto the other.

21. Draw one triangle and rotate it 180° about any point. How many reflections are required to map the first triangle onto the second?

22. $\triangle XYZ$ uses points $X(0,0)$, $Y(4,3)$, and $Z(5,2)$. Suppose $\triangle XYZ$ is reflected to get $\triangle X_1Y_1Z_1$, where $X_1 = (0,0)$, $Y_1 = (3,4)$, and $Z_1 = (2,5)$. Precisely describe the line of reflection (with an equation or some other way).

23. Suppose the three points $X(0,0)$, $Y(4,3)$, and $Z(5,2)$ are each rotated 90° counterclockwise around the origin to obtain $\triangle X_1Y_1Z_1$.

a. Find the coordinates of X_1, Y_1, and Z_1.

b. Find some lines so that reflecting across the lines is the same thing as rotating 90° counterclockwise.

A RIGHT TRIANGLE DISSECTION

THE INVESTIGATION

Show that it's possible to divide a right triangle into n^2 congruent triangles, for any whole number n. For example, a right triangle can be divided into 5^2 or 25 congruent triangles.

TAKE IT FURTHER

24. Will your process for division of right triangles work for all triangles (not just right triangles)? Explain.

25. What is the relationship between the area of the original undivided right triangle and the area of one of the small triangles?

26. What is the relationship between the perimeter of the original undivided right triangle and the perimeter of one of the small triangles?

CONNECTING MIDPOINTS ON A TETRAHEDRON

THE INVESTIGATION

Construction paper works well. So do toothpicks and gumdrops. If you use the latter, prepare for the midlines by making each side of the tetrahedron two toothpicks long, with a gumdrop at the midpoint.

Build a model of a regular tetrahedron. On each face, construct all the midlines. You get four small tetrahedron "caps" at each corner. If these are cut off, what kind of solid remains? Try to *picture* this construction in your head: Connect the midpoints of the sides of a cube, then cut off the corners. Ask yourself questions like these:

- What kind of shape gets cut off at the corners?
- How many new faces do I create by slicing off the corners of the cube?
- What shape are these new faces?
- Of the original faces, what remains (what shape)?

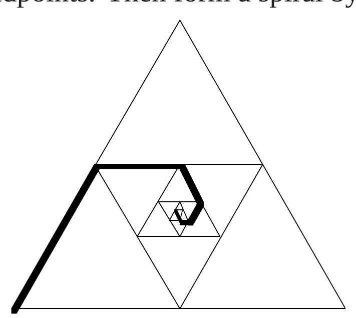
Sketch, describe with words, or make a model of the shape you get from this construction.

TAKE IT FURTHER.................................

27. Investigate what kinds of solids are formed when midpoints of edges of some other solid are connected and then the corners are sliced off, as you did with the tetrahedron. Are the faces of the new solid equilateral? Triangular? Congruent? If you start with a regular polyhedron, will the new solid also be regular? Illustrate your answers with sketches or models.

SPIRAL

Make a series of nested equilateral triangles, like the picture here. Start with a big triangle and connect the midpoints. Then form a spiral by connecting midpoints.



28. Draw this spiral (to a depth of six “arms”) on paper or on your computer. How long is the spiral?

29. How long would the spiral be if it had ten arms? If it had 100 arms?

30. As more triangles are drawn and the number of arms gets bigger, what happens to the length of the spiral? How long can the spiral get?

CONGRUENCE FROM PARTS

THE INVESTIGATION

The lengths of the three sides completely determine the size and shape of a triangle (SSS). Do the lengths of the three *midlines* uniquely determine a triangle? That is, could there be a midline-midline-midline (MMM) congruence postulate?

Use hand or computer tools.

- Construct a triangle in which the three midlines have these lengths: 1″, 3″, and 2.5″. Can you make more than one triangle with these same lengths for midlines?
- If two triangles have the same length midlines, are the triangles congruent? Give a proof or a counterexample.

Do the lengths of three *altitudes* completely determine the size and shape of a triangle? That is, is there an "Alt-Alt-Alt" triangle congruence test?

- Construct a triangle in which the three altitudes have these lengths: 4″, 3″, 2.75″. Can you make more than one triangle with these same lengths for altitudes?
- If two triangles have the same length altitudes, are the two triangles congruent? Justify your answer with evidence and logical argument.

Can *any* three lengths be the altitudes of a triangle?

31. Write and Reflect Drawing a triangle from a given set of altitudes is quite a challenging problem. Write instructions that tell how to construct a triangle from any set of three altitudes.

TAKE IT FURTHER

32. Is there a "Median-Median-Median" test for triangle congruence?

33. Is there an "Angle bisector-Angle bisector-Angle bisector" test for triangle congruence?

MAKING QUADRILATERALS FROM CONGRUENT TRIANGLES

THE INVESTIGATION

To prepare for the investigation:

1. Carefully construct and cut out the following sets of congruent triangles:
   - four triangles with sides 3″, 4″, and 6″;
   - four right triangles with sides 3″, 4″, and 5″;
   - four isosceles triangles with sides 3″, 3″, 4″.

2. The triangles in the first two sets have a “front” and a “back.” For each set, decide what you will call the “front,” and then color that side red. (Leave the back uncolored.) Also color one side of each of the isosceles triangles red.

Pick two congruent triangles and put them together to form a quadrilateral.

1. Draw all the different all-red quadrilaterals you can form from your chosen triangles.

2. Draw all the different mixed-color quadrilaterals that you can form.

What kind of quadrilaterals have you found? Note the type of triangles that you used to make them. Describe any special characteristics (congruent or parallel sides, diagonals that are congruent, are perpendicular, or that bisect each other, . . .) that these quadrilaterals have.

Repeat the experiment using sets of four congruent triangles, first all-red, then mixed-color quadrilaterals. What shapes are created this way? Record your findings in a similar way.

Summarize your results in a chart or poster.

TAKE IT FURTHER

34. Use the language of rotation and reflection to describe how each type of quadrilateral could be formed from the congruent triangles.

35. Identify the common characteristics of the all-red quadrilaterals and the mixed-color quadrilaterals.

Investigation 2.12

MYSTERY FIGURES

By now, you have learned a lot about how to tell if triangles are congruent and how to use ideas of congruent triangles in mathematical proof. But the world is not made up of triangles. How can you tell if other two-dimensional shapes are congruent? What tests do you need to make? Can the tests for triangles help? And what about solid shapes like cubes, spheres, and cones? These are just some of the questions you will examine in the following investigations.

Where Are the Drawings? Each problem describes a figure and makes a true statement about it. You have two mysteries to solve for each problem.

- What does the figure look like?
- Why is the statement true?

Most of these challenges in drawing, conjecturing, and proving can be approached in more than one way.

Directions Follow the general directions for all the mystery problems:

- Make a careful drawing of the figure that is described.
- Using your drawing and your own words, explain what the statement says about the figure.
- Write what you know about the figure. Make conjectures about things you think might be true, and explain why.
- Try to prove the statement using any style of proof you like.

MYSTERY PROBLEMS

1. From $\square ABCD$, extend $\overline{AB}$ through B to point E, extend $\overline{BC}$ through C to F, extend $\overline{CD}$ through D to G, and extend $\overline{DA}$ through A to point H, so that $BE = CF = DG = AH$. Quadrilateral $EFGH$ is a square.

Some of these problems, such as this one, are particularly good to do with geometry software.

2. On a given isosceles triangle, pick any point along the base, and draw segments parallel to the congruent sides, forming a parallelogram. The perimeter of the parallelogram is fixed, regardless of which point is picked along the base.

3. On sides $\overline{BC}$ and $\overline{AC}$ of any triangle ABC, construct equilateral triangles BCD and CAE outside of the original triangle. Then $BE = AD$.

The *foot* of the altitude is the point at which it meets the base.

4. An altitude is drawn to the hypotenuse of a right triangle. Segments drawn from the foot of the altitude to the midpoints of the legs are perpendicular to each other.

5. One of the diagonals of a parallelogram is trisected. Segments joining these trisection points to the midpoints of a pair of opposite sides form a parallelogram.

6. P is any point in the base, $\overline{BC}$, of an isosceles triangle ABC. The side $\overline{AC}$ is extended through C to E so that $CE = CP$. $\overline{EP}$ is drawn and extended to meet $\overline{AB}$ at F. Then, $m\angle EFA = 3m\angle AEF$.

7. If the diagonals of a quadrilateral are congruent and also one pair of opposite sides are congruent, then at least one of the triangles into which the quadrilateral is divided by the diagonals is isosceles.

8. $\triangle\ ABC$ is isosceles, with $AB = BC$, and with W as the midpoint of $\overline{BC}$. A line from W perpendicular to the base $\overline{AC}$, meets $\overline{AC}$ at K. Prove that $KC = \frac{1}{4}AC$.

9. On a quadrilateral with no two sides congruent or parallel, the lines that connect the midpoints of its opposite sides bisect each other.

Investigation 2.13

BEYOND TRIANGLES

Even if triangles are the simplest polygon, quadrilaterals may be the easiest to find in the real world. Quadrilaterals are everywhere: in street maps, window panes, books, packaging, and so on.

You probably know a great deal about quadrilaterals: from the earliest elementary grades, you've constructed, measured, cut, and folded squares and rectangles. Use this practical experience now as you begin a more formal study of quadrilaterals. While the focus is on congruence, all of the properties of these shapes are important in understanding what makes each one unique and what factors determine their congruence.

FOR DISCUSSION

Here is a list of several quadrilaterals: trapezoid, square, rectangle, parallelogram, rhombus, kite.

Do all of these quadrilaterals appear regularly in the world around you? Make a list of several "real-life" examples of each type of quadrilateral. Find an example of each in your school.

Drawings and photographs contain more trapezoids than the 3D objects they depict. Do they help you find good examples of other quadrilaterals?

Show what you know of the quadrilateral types by constructing an accurate drawing for each figure described below. As you draw each figure:

- Review the definition of that type of quadrilateral.
- Study the figure and list additional properties that appear to be true. (In thinking about additional properties, it can help to show diagonals in the figure.)
- Compare your drawing to those made by other students, and decide whether the given set of directions describes a unique figure or many possible correct figures.

1. Construct a parallelogram with one pair of sides measuring 5 cm and one pair of sides measuring 10 cm.

2. Construct a parallelogram with one pair of opposite sides measuring 10 cm and one angle measuring 45°.

3. Construct a rectangle with a 3″ diagonal and a 1″ side.

4. Construct a rectangle with diagonals meeting at a 120° angle and a side 6 cm long.

5. Construct a rhombus with a 4″ side and a 2″ diagonal.

6. Construct a rhombus with diagonals measuring 3″ and 2″.

7. Construct a kite with a 6 cm side and an 8 cm side, with the angle between them measuring 120°.

8. Construct a trapezoid with two right angles, legs measuring 5 cm and 6 cm, and one base of length 10 cm.

9. Construct a parallelogram with a side 6 cm, the length of a diagonal 12 cm, and the altitude to the given side 8 cm.

10. Construct a square with a diagonal measuring 2″.

This problem requires a considerable amount of work, and your results will be important. Work with others to share the work and results.

11. For each type of quadrilateral, list the special properties it *appears* to have. Then, for each property, find a convincing argument to show why it *must* be true for *all* quadrilaterals of that type. For example, if you noted that opposite sides are the same length in one of your parallelograms, try to prove that this will be true for all parallelograms.

You might, for example, imagine that you're writing a book about quadrilaterals and deciding which ones could be used as chapter titles, and which ones might fit best as subsections of a chapter.

12. **Write and Reflect** Make a map, flow chart, outline, family tree, or other system to classify all the quadrilaterals, and to show how they're related and what features they share. If you make a map, think about which quadrilaterals belong in each region and how the regions are connected. Use both logic and imagination in designing your diagram. There are many correct ways to do this.

TO BE SURE OR NOT TO BE SURE . . .

A carpenter accurately cuts four boards to frame a door: two sides of 80″ (on the edge that fits against the door), and a top and bottom of 30″ (again at the edge by the door). Using a carpenter's square, one side piece is set at a right angle to the floor piece. This guarantees that the frame has one right angle and opposite sides of equal length. Is

that enough to ensure that the *other* three corners are right angles, and, therefore, that it will fit a rectangular door?

Judging the shape of a quadrilateral from incomplete information is not always easy, but your intuition is often correct. In each problem below, decide whether the information given is sufficient to be sure of the shape of the figure. You may answer from intuition, but then hold yourself to more rigorous standards of logical argument in your explanations.

13. Complete the problem of the carpenter and the door. Is the information given sufficient to determine that the frame is rectangular? Write a convincing argument or proof.

14. Is a quadrilateral with congruent opposite sides guaranteed to be a parallelogram? Write a convincing argument.

15. If a quadrilateral's diagonals bisect each other, what shape must it be? Write a convincing argument.

16. If a quadrilateral has one pair of opposite sides parallel and the same length, must the other pair of sides also be parallel? Write a convincing argument.

17. A quadrilateral with four congruent sides is a rhombus, by definition. Find some other set of information that would be enough to prove that a quadrilateral is a rhombus.

FOR DISCUSSION

What ways do you know to prove that a figure is a parallelogram? For example, will information just about angles (or just about diagonals, or . . .) be sufficient to prove that a quadrilateral is a parallelogram, or must you also know something about sides?

APPLICATIONS OF ANALYSIS AND PROOF

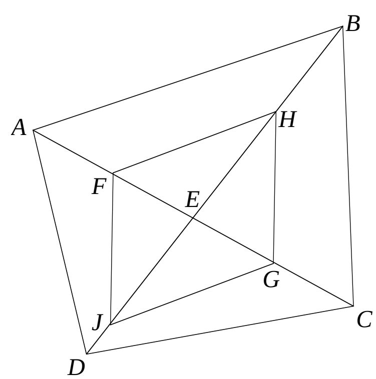
Quadrilaterals are more complicated than triangles. They have diagonals, more sides, and more angles. Greater complexity means greater difficulty in proofs and problems. As you work, think of conjecture, analysis, and proof as three aspects of the same problem-solving process. But remember, they won't always occur in the same order. Sometimes the conjecture will appear first, and you will try to prove it. Sometimes an analysis for the proof of one conjecture will suggest a new conjecture.

Originally, the authors made a mistake and didn't catch it in early drafts of this module! For *some* rectangles, the angle bisectors *don't* create a quadrilateral! Can you discover which ones?

18. In a triangle, the angle bisectors are concurrent. In many rectangles, the angle bisectors intersect to form a quadrilateral. What type of quadrilateral? Support your conjecture.

19. What type of figure is formed by the angle bisectors of other types of quadrilaterals? Write up your conjectures, including sketches and justification.

20. In quadrilateral *ABCD*, $AF = FE = EG = GC$ and $JE = EH$. Also, $\overline{AC}$ bisects $\angle DAB$ and $\angle BCD$. What conjectures can you make about the characteristics of *ABCD* and *FHGJ*? Write proofs to support your conjectures.



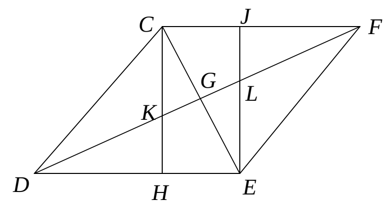
21. What type of quadrilateral is formed by connecting the midpoints of a rectangle's sides? Prove your conjecture.

22. In rhombus *CDEF*, $\overline{CH}$ and $\overline{JE}$ are altitudes.

a. What type of quadrilateral is *CJEH*? Prove it.

b. Prove that $\triangle GLE \cong \triangle GKC$.

c. What other triangles can be proven congruent?

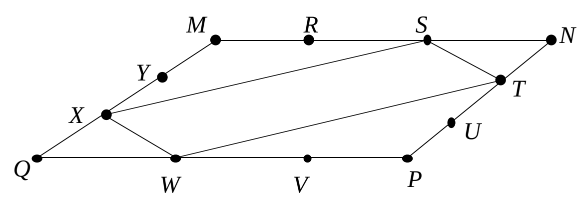


23. The sides of parallelogram *MNPQ* have been trisected; four of those points of trisection have been connected to form quadrilateral *STWX*.

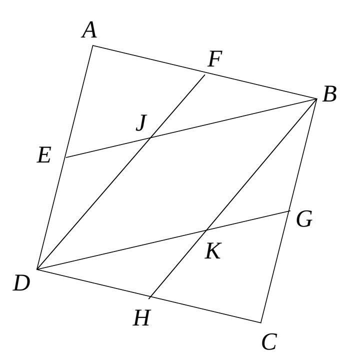
a. List some things that can be proved about the figure described.

b. What kind of quadrilateral is *STWX*? Prove your conjecture.

c. Which of your results could you still prove if *MNPQ* were not known to be a parallelogram?



24. Two opposite vertices of a square (B and D) are connected to the midpoints of the opposite sides.



a. Name all the quadrilaterals that are formed.

b. Conjecture about their shapes. Prove what you say. (Modify your conjecture if necessary.)

The computer can help you study these varying shapes.

c. Vary the shape of *ABCD* and study the relationships between the inner quadrilaterals and the shape of *ABCD*. For example, you might investigate the shape of *JBKD* when *ABCD* is a kite, or you might see whether it is possible to make *JBKD* a square.

d. Write up the results of your investigation. Include sketches, conjectures, and proofs.

PROVING CONGRUENCE FOR QUADRILATERALS

Each of the recent problems has focused on the properties of *one* quadrilateral. In working the problems, you probably needed to use congruent triangles more than once as a justification or explanation. How does one prove *two* quadrilaterals are congruent to each other? What determines congruence for two quadrilaterals?

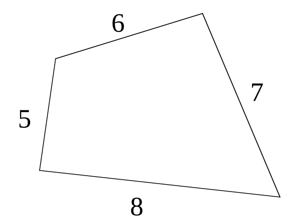
The basic definition of congruence for polygons still applies. Two quadrilaterals are congruent if all corresponding sides and angles are congruent.

FOR DISCUSSION

Quadrilateral "parts" include diagonals as well as sides and angles.

For triangles, the congruence postulates are SSS, SAS, AAS and ASA. Information about three parts can totally determine the shape of a triangle and is, therefore, the basis of each congruence postulate. How many parts must be known to fully determine the shape of a quadrilateral?

25. For polygons with more than three sides, it's more difficult to be sure that the sides and angles are "corresponding." For example, here is one order in which segments of lengths 5, 6, 7, and 8 can be arranged to form a quadrilateral. How many other orders are there?



26. **SSSS?** Is there a quadrilateral congruence test equivalent to SSS? That is, if two quadrilaterals have congruent corresponding sides, are they congruent?

27. Is SAS a congruence test for quadrilaterals? That is, will knowledge about two sides and the included angle fully determine the shape and size of a quadrilateral? Develop a quadrilateral version of SAS: a congruence test for quadrilaterals that uses sides and included angles.

28. Is ASA a congruence test for quadrilaterals? Develop a quadrilateral version of ASA: a congruence test that uses angles and included sides.

29. The congruence tests you have been working on are for all quadrilaterals. Develop a congruence test that will work specifically for rectangles. The goal is to find the minimum information that will completely determine the shape of a rectangle, or will show that two rectangles are exactly the same size and shape.

30. Each congruence rule below is incomplete. Add as little information as you can to make each a true test for congruence for that type of quadrilateral.

a. Two parallelograms are congruent if they have congruent diagonals and . . .

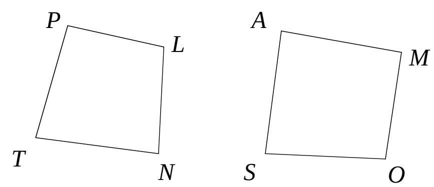
b. Two rhombi are congruent if they have one pair of corresponding sides congruent and . . .

c. Two trapezoids are congruent if they have the same base angles and height (height is measured between the parallel sides) and . . .

In a trapezoid, how many angles do you need to know to know all the angles?

CHECKPOINT

31. Here are two quadrilaterals. Decide whether each set of given information is sufficient to prove that the two quadrilaterals are congruent. Justify your decision. (Note that the figures are *not* drawn to scale.)

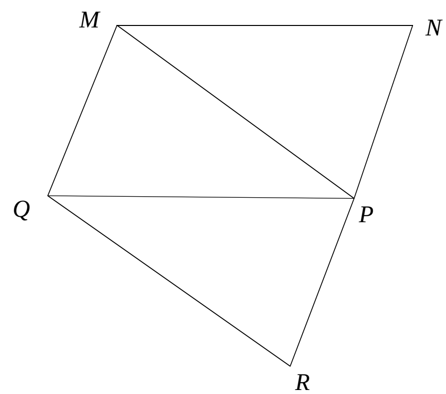


a. Is it correct to conclude that $PLNT \cong AMOS$ if:
- $PT = AS$?
- $LN = MO$?
- $\overline{PT} \parallel \overline{LN}$?
- $\overline{AS} \parallel \overline{MO}$?

b. Is it correct to conclude that $PLNT \cong AMOS$ if:
- $\angle P$ and $\angle A$ have the same measure?
- $\angle N$ and $\angle O$ have the same measure?
- $TN = SO$?

c. Is it correct to conclude that $PLNT \cong AMOS$ if:
- $PLNT$ is equilateral?
- $AMOS$ is equilateral?
- $PT = AS$?

32. *MNPQ* is a parallelogram and *MPRQ* is a parallelogram.



a. Are the two parallelograms congruent? Explain.

b. What congruent triangles are there in the figure? Justify each choice.

c. As drawn, $\overline{NPR}$ looks straight. From the information you have (*MNPQ* and *MPRQ* are both parallelograms), is it possible to prove that *N*, *P*, and *R* are collinear? Explain.

d. If $\angle N$ measures 70 degrees, what other angle measures can be determined?

e. If $MN = 4.0$ cm, and $PR = 2.5$ cm, then what is the perimeter of *MNPRQ*?

TAKE IT FURTHER. . . . .

33. If you know the lengths of the sides of a particular polygon, how many diagonal lengths do you need to know to be able to construct a congruent copy of the polygon? Develop a congruence rule for quadrilaterals and other polygons that uses sides and diagonals (but no angles). Be specific about how many diagonals must be known, and about which diagonals must be known.

Investigation 2.14

CONGRUENCE IN THREE DIMENSIONS

Extend your ideas about congruence from two to three dimensions. Try to answer each question just by visualizing and creating images in your mind. After you have made a mental picture, make a model or sketch to illustrate your thinking.

1. **Write and Reflect** What does it mean for two three-dimensional figures to be congruent?

2. Are two spheres that have the same volume congruent?

   a. What measurements are needed to guarantee congruence in two spheres?

   b. Do congruent spheres have the same surface area?

3. Imagine two rectangular solids (boxes) that are 12 inches tall and 5 inches wide. Are they congruent?

4. Imagine two empty boxes. Both are 10 inches tall and both contain exactly the same volume of air. Are the two boxes congruent?

5. Imagine two noncongruent cylinders that have the same height. Name three measurements that could be taken for the cylinders that would be different. Are there any measurements (besides height) that could be the same, even if they are not congruent?

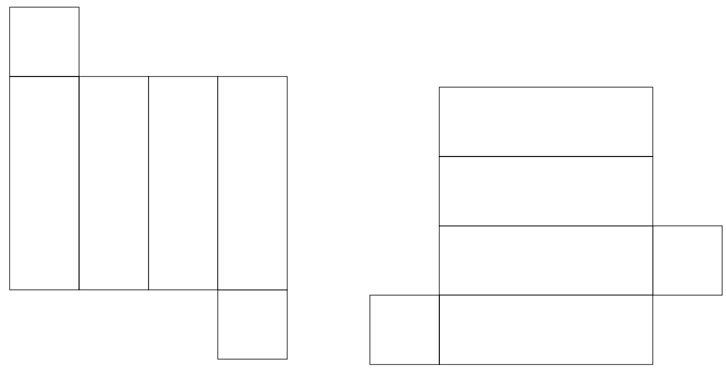
6. Are all cubes congruent? What is the minimum number of measurements required to insure congruence for two cubes?

7. A tetrahedron has four triangular faces. Imagine a tetrahedron whose faces are all equilateral. How many edges does it have? How many different edge *lengths* are there? Picture all of the angles that could be measured for this solid. How many different angle sizes are there?

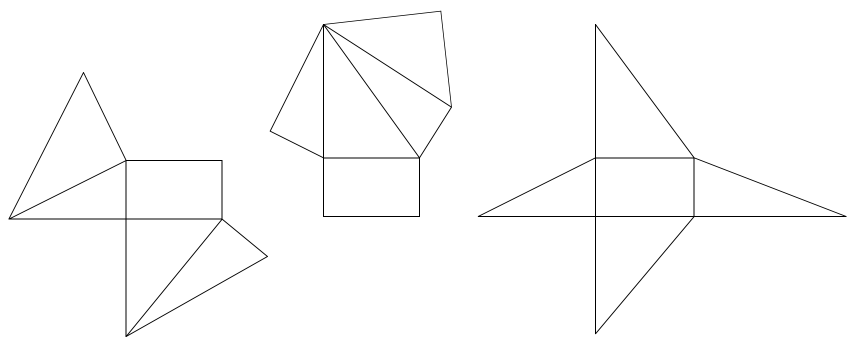
8. Define a "box" as a three-dimensional object with six faces, all of them quadrilaterals (but not necessarily rectangles or even parallelograms). Design a way to test two boxes for congruence without measuring any of the angles.

You might find it interesting to enlarge these on a photocopy machine and use them to create three-dimensional models.

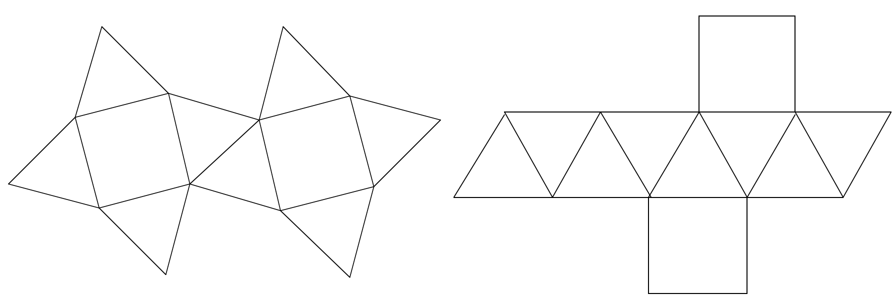
9. Following are four sets of unfolded solid objects (these are called *nets*). For each set, imagine how they could be refolded and decide if the objects in each set would fold to form congruent solids.



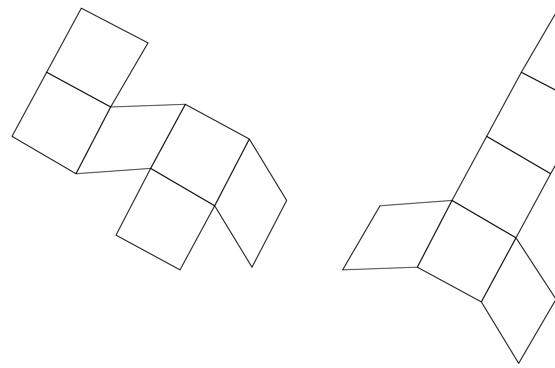
Set A



Set B



Set C



Set D

10. Draw two different nets that will fold to make congruent triangular prisms.

11. It is possible to create two noncongruent polygons from the same set of (more than three) edges. Is it possible to create two noncongruent solids from the same set of faces? Try to picture these in your mind.

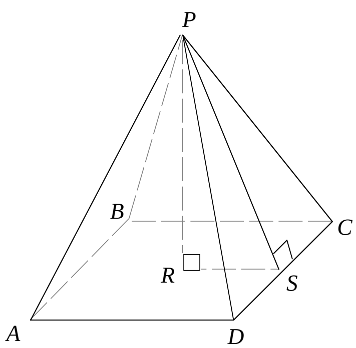
FOR DISCUSSION

Congruent figures have congruent corresponding parts. For triangles, the "parts" were angles and sides. What are the parts that must be considered in testing solids for congruence? Before discussing this question, it might help to list the solids you know and their basic properties.

12. Some of these statements contain too much information. Rewrite each statement leaving out anything that is extra.

a. Two cubes are congruent if they have the same surface area and height.

b. Two rectangular solids are congruent if they have the same volume, length, width, and height.



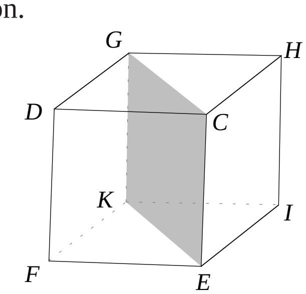
A square right pyramid: *ABCD* is the square base; *PR* is the height; *PS* is the slant height.

c. Two square right pyramids are congruent if they have the same slant height, height, and base area.

d. Two cylinders are congruent if they have the same height, circumference, and surface area.

e. Two regular tetrahedra are congruent if they have the same height and volume.

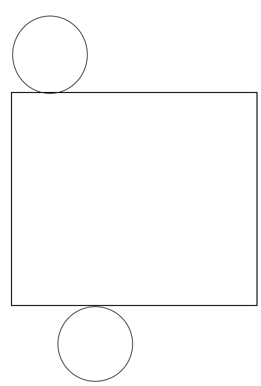
13. a. In the figure below, a plane intersects a cube at opposite vertices. If the plane cuts the cube into two solids in this way, will the two solids be congruent? Justify your decision.



b. There is more than one way for a plane to intersect a cube only at midpoints of its edges. Sketch the different ways. For which cuts (if any) are the two parts congruent solids?

c. If a plane divides a cube into two congruent solids, will these solids have equal volumes? Will they have equal surface areas?

Some solids, like the cube, are made of plane surfaces called faces. Other solids, like cylinders and spheres, have curved surfaces. Of the solids that are built of curved surfaces, some can be built from plane figures, suitably bent. A plane figure for such a solid or surface is called its *development*. Here, for example, is the development for a cylinder.



TAKE IT FURTHER

14. Copy one of the nets for the rectangular pyramid from Problem 9. Use tick marks to indicate all of the segments that *must* be congruent for the net to fold into the pyramid.

15. In the development of a cylinder, indicate what measurements must be equal for the development to "work."

16. Draw two different developments that will make congruent cylinders.

17. Some solids built of curved surfaces can *not* be developed from plane figures. For each of the following, either show the plane figures from which it can be built, or say what makes you believe it cannot be built from plane figures.

- **a.** sphere
- **b.** cone
- **c.** the bell (horn part) of a trumpet
- **d.** the body of an acoustic (nonelectric) guitar
- **e.** a football
- **f.** a soup bowl
- **g.** a playground slide

Module Overview and Planning Guides

About the Module T2

Main Timeline Planning Chart T2

Alternate Timelines T6

Assessment Planning T8

ABOUT THE MODULE

This module focuses on the mathematical topics of congruence and proof. The module is problem-based. You will notice that the notes in this guide often refer to specific problems and the outcomes from students' investigations of the problems. The habits of mind focused on in this module include explaining and proving, and using formal and informal mathematical language to describe things.

The study of congruence begins in an investigatory way and moves into formal proof. The congruence postulates for triangles are discovered experimentally. Formal proof is introduced as we use the postulates as assumptions to prove triangles congruent and we use corresponding parts of congruent triangles to prove other results. Students look at different ways to present formal proofs and learn about the difference between *coming up with* a proof and *writing up* a proof.

This module is ideal for:

- high-school geometry
- middle-school geometry (use the "Investigatory" alternate timeline)
- a course for preservice teachers

MAIN TIMELINE PLANNING CHART

Using all the investigations in this module will take an average class 8 to 10 weeks. Please note that Investigations 2.11 and 2.12 are meant as flexible resource activities. Plan to pull problems and activities from them regularly.

Investigation	Description	Key Content	Pacing
2.1 Polyominoes	This is a "hands-on" puzzle-like activity for pairs or small groups. Students design polyomino shapes, combining them to form larger congruent shapes.	• meaning of *same shape* • combinatorics	1–3 days

MODULE OVERVIEW AND PLANNING GUIDES
(CONTINUED)

Investigation	Description	Key Content	Pacing
2.2 Comparing Pictures	This is a full-class, get-to-know-you activity. Students find partners by matching pictures of common objects. Discuss how size, shape, and orientation of figure relate to congruence.	• defining congruence • reflection, rotation	2 days
2.3 The Congruence Relationship	This investigation includes reading and problems. The congruence symbol and congruence notation are introduced. Main ideas: superimposability, equal measurement determines congruence of line segments and angles.	• congruence notation • corresponding parts	2 days
2.4 Strong Language	This investigation includes reading and class discussion. Students distinguish between statements about particular figures and generalizations about whole classes of figures.	• concept of theorem	partial day
2.5 Triangle Congruence	This investigation centers around a small-group activity, "The Envelope Game." Students construct triangles from sets of parts chosen randomly from an envelope. Class discussion: How much information is enough to fully determine the triangle shape and to prove congruence?	• SSS, SAS, ASA • counterexample	3–4 days
2.6 Warm-Ups for Proof	Students prepare convincing arguments about a variety of situations.	• introduction to proof	1 day
2.7 Writing Proofs	Proof is explained formally. Students read and write congruent triangle proofs in two-column, paragraph, and outline styles.	• experiment vs. deduction • Why prove things? • triangle congruence proof	3 days
2.8 Analysis and Proof, Part 1	Students read, discuss, and explain. Students learn how to analyze or come up with proofs. More congruent triangle proofs are provided.	• CPCTC (Corresponding parts of congruent triangles are congruent.) • flow charts • logical analysis	2–3 days

Investigation	Description	Key Content	Pacing
2.9 Analysis and Proof, Part 2	Students come up with proofs by starting at the conclusion and working backward, asking “What do I need to prove that?” Students identify hypothesis and conclusion and write statements in “if/then” form.	• analysis by need	2–3 days
2.10 Perspectives on Proof	Included for student reading and discussion are two mathematicians’ essays on proof.	• hypothesis and conclusion • mathematicians on proof	1 day
2.11 Investigations and Demonstrations	Eight problems suitable for exploration and assessment are presented.		
Perpendicular Bisectors	Proof of theorem: Points on perpendicular bisector of segment are equidistant from the segment’s endpoints. Included are constructions, paper folding, and computer investigations about perpendicular bisectors of sides of a triangle and of chords in a circle.	• perpendicular bisectors • chords in a circle	2 days
Angles and Sides in Triangles	Students investigate side and angle relationships in isosceles and scalene triangles. Understanding of SSA is refined.	• isosceles, scalene • hypotenuse leg (HL) • congruence test • SSA (non)congruence	2 days
Isosceles Triangle Proofs	Students demonstrate that any two of four statements about isosceles triangles will prove the others. Six proofs are included.	• isosceles triangles • medians, altitudes • angle bisectors	2 days
Reflection Puzzle	Students look for the smallest number of reflections required to “move” a triangle to coincide with a congruent copy located in some random place on the plane. This activity can be done in or out of the computer lab.	• reflections	2 days
A Right Triangle Dissection	Students find methods for dividing a right triangle into n congruent triangles and discuss what values of n are possible.	• dissection • right triangles • area and perimeter	1 day

MODULE OVERVIEW AND PLANNING GUIDES
(CONTINUED)

Investigation	Description	Key Content	Pacing
Connecting Midpoints on a Tetrahedron	Students build a model of a tetrahedron and then slice off its corners to form a new solid. Cubes and other solids are also investigated.	• tetrahedron • spirals • cross sections	2 days
Congruence from Parts	Is there an MMM (midline-midline-midline) congruence postulate for triangles? Altitudes (AltAltAlt) are also investigated as a basis for a congruence test.	• congruence postulates • altitudes and midlines	2–3 days
Making Quadri-laterals from Congruent Triangles	Students use cardboard triangles to construct quadrilaterals. Then students categorize the quadrilaterals according to whether they had to slide, flip, or rotate one of the triangles.	• quadrilateral properties • reflections, rotations	1 day
2.12 Mystery Figures	Students are challenged to draw complex figures from a written description and then to explain or prove statements about the figures. This material is suitable as classwork, homework, assessment, or challenging extensions.	• drawing from directions • congruence • challenge problems	resource
2.13 Beyond Triangles	Students use constructions and concept maps as background for developing a set of congruence rules for quadrilaterals.	• quadrilateral properties • congruence • necessary and sufficient	4–5 days
2.14 Congruence in Three Dimensions	Students visualize and construct 3-dimensional objects and their nets and then consider rules for determining congruence.	• congruence of solids • nets and developments • volume, surface area	2–3 days

ALTERNATE TIMELINES

We offer here three alternative paths through the module, each with a specific emphasis and all considerably shorter than the main timeline plan. If the full 7–9 weeks is too long for your class, we hope one of these will suit you. These three alternatives were chosen to meet the most frequent requests of field-test teachers.

Concept-Focused
- **2.2–2.3 (5 days)**
- **2.5 (3 days)**
- **2.7 (3 days)**
- **2.8–2.9 (5 days)**
- **2.13 (4 days)**
- **2.11 (3 days)**

Short, Concept-Focused (4–5 weeks)

This is a good plan to follow if you want to cover the whole module but don't have 7–9 weeks. Spend just four or five weeks working through it; stress concepts, not pressing for maximum difficulty on proofs. Choose just one type of analysis tool and style of proof from the several offered in the Student Module. For assessment, have each student group present one of the investigations in Investigation 2.11.

Proof-Focused
- **2.2 (2 days)**
- **2.3 (2 days)**
- **2.4 (1 day)**
- **2.6–2.7 (4 days)**
- **2.8–2.9 (5 days)**
- **2.11 (first three investigations) (4 days)**
- **2.10 (1 day)**
- **2.11 "Congruence from Parts" (3 days)**
- **2.13 or 2.14 (5 days)**

Focus on Proof (6 weeks)

This timeline is for those who wish to take the fullest possible advantage of the module's deep and wide offerings on proof while de-emphasizing the hands-on activities.

If your class is familiar with the notion of congruence, you can skip "Comparing Pictures" (2.2), and jump right into "The Congruence Relationship" (2.3), with just a brief introduction. (You may choose Problems 5–9 from Investigation 2.2 to serve as that introduction, or you can use an activity of your own.) Also, you can assign many of the Problems from Investigations 2.3–2.5 for homework, decreasing the class time from the estimates shown here.

In "Triangle Congruence" (2.5), omit the first envelope game, but do all of the "Take It Further" problems. Problems from "Mystery Figures" (2.12) can be used as a resource for in-class work beginning when students are learning to analyze proofs, and they can also be used as assessment. In Investigation 2.11, "Congruence from Parts" also makes a good final assessment for the whole module.

This timeline suggests ending with either "Beyond Triangles" (2.13) or "Congruence in Three Dimensions" (2.14) to keep students thinking about altering problems and extending their knowledge.

Investigatory

- 2.1 (2 days)
- 2.2–2.3 (5 days)
- 2.5 (3 days)
- 2.11 (10 days)
- 2.13 (4 days)

Mostly Investigatory (4–5 weeks)

This timeline highlights hands-on work and explorations and de-emphasizes formal proof. Students still make logical arguments. It is ideal for middle-school mathematics classes studying geometry or for other classes where proof is not a major focus.

Two weeks are given to the investigations in Investigation 2.11. As written, some of these focus on proof; alternatively, you can encourage students to explore and conjecture, ending with reasoned arguments and explanations that may not look like proofs. Here are some specific ideas:

The 2.11 investigations (A Reflection Puzzle, A Right Triangle Dissection, Connecting Midpoints in a Tetrahedron, and Making Quadrilaterals from Congruent Triangles) are investigatory in nature and can be used as written.

Perpendicular Bisectors: Rather than proving the theorem, have students construct the perpendicular bisector of a segment (on paper or with software, making sure it is a careful construction and not a drawing) and several points (or one moveable point in software) on that line. They can connect the points to each endpoint and measure (or fold to compare the lengths), and develop the conjecture that every point on the line will be equidistant from the two endpoints. Encourage students to look for an explanation based on triangle congruence. Then assign Problems 4–8 (in Problem 7, ask for an explanation rather than a proof) as further explorations.

Isosceles Triangle Proofs: Again, assign this as a software or paper-and-pencil investigation rather than as a proof. Ask students to construct something that meets two of the three criteria and decide if it also meets the other two. Work on arguments based on triangle congruence. Then choose one or more of the “Take it Further” problems to investigate.

Congruence from Parts: As written, this investigation does not require any proof. Be warned, though, that some of the constructions are quite difficult. This is a very challenging investigation!

ASSESSMENT PLANNING

What to Assess

- The student can define congruence, interpret statements about congruent figures, test for congruence in triangles, and prove that two triangles are congruent.
- The student understands the difference between verifying a statement by experiment and proving the statement, and can use both techniques.
- The student has developed a working vocabulary about triangles, quadrilaterals, and other polygons and solids.
- The student can make reasonable conjectures, explore them, and defend or reject them with logical argument.

Notebooks

Throughout the entire module, we recommend that students keep a notebook containing:

- daily homework and other written assignments
- a list of vocabulary, definitions and theorems that emerge during classwork and homework

Projects

At some time near the end of the module, have students complete a project and present it to the class. Suggestions:

- Complete one of the investigations in 2.11.
- Draw and prove a Mystery Figure problem from Investigation 2.12.
- Complete a computer lab exploration.

QUIZZES AND JOURNAL ENTRIES

Investigation	Journal Suggestion or Presentation	Quiz Suggestion
2.3	*Write and Reflect* Problem 1: Explain how to tell if two objects are congruent.	See Teaching Notes for ideas.
2.5	Problem 29: Divide a triangle into congruent pieces and justify solution.	
2.6	Problem 4 or 5: Write a convincing argument.	
2.7	Problems 2 and 3: Explain experimental and deductive proof of vertical angle theorem.	See Teaching Notes for ideas.
2.8	Write up an analysis of a proof from the “Checkpoint.” Challenge: Do Problem 16.	
2.9	Analyze a proof more than one way, or find a proof that previously seemed too difficult and show how to analyze it.	See Teaching Notes for ideas.
2.10	Write about one of the essays. See suggestions in Teaching Notes for specific assignments.	
2.13	*Write and Reflect* Problem 12: Write a convincing argument.	

POLYOMINOES

The first five investigations of the module introduce students to ideas of congruence, first informally and then with a formal definition, and then to the triangle congruence postulates. Students begin by comparing pictures, and should return often to the notion that two figures are congruent if one fits exactly on top of the other. SSS, SAS, AAS, and ASA are discovered by students through experiment, and remain postulates without proof. This section ends with students deciding if triangles are congruent and providing justification based on the congruence postulates. This sets the stage nicely for exploring more formal proof in the next section, "Investigating Proof."

OVERVIEW

Materials:
- **graph paper**
- **scissors**
- **tape**
- **tetrominoes**

The game-like aspect of this introductory investigation makes it a fun way to begin the study of congruence. In solving a set of polyomino puzzles, students discuss what it means to say that two shapes are the same. Students try to find out how many different polyominoes there are for a given number of squares. Later, they study ways that the polyominoes can be combined to form larger figures. Besides being very engaging, the activity promotes visualization skills, systematic approaches to problem solving, and an informal introduction to the vocabulary of transformations, shape, and congruence. In some classes, the problems may spark discussion of questions in combinatorics.

Definition:
Two polyominoes are considered the same if one could be placed on the other so that they coincide.

The polyomino investigations work best when students work with partners or in a small group. Assume that most students will need to cut out the shapes and manipulate them to decide if they are the same or different. Graph paper works well to simplify sketching the various polyominoes, but students should be reminded to use squares larger than $\frac{1}{4}$ inch for anything they intend to cut out.

TEACHING THE INVESTIGATION

Alternate beginning:
Have students read and do the first two problems for homework; then use class time to work on the more difficult problems.

Begin the class with a discussion of what polyominoes are and then ask the students to show that all the different figures can be made from two squares and then three squares.

To begin the tetrominoes problem, ask students to *visualize* different ways that four squares could be combined. Then, have the students work in small groups to complete the problem using sketches and cutouts for testing. Ask one group to report its results to the whole class. If other groups challenge these results, have them explain why they think they are right.

The remainder of the class can be spent with groups working on problems. Allow time for presentations and arguments about the results. Students will find many different solutions and should be encouraged to describe their results.

Some good questions to ask as students work or present their results:

- How do you know that you have found *all* the ways to arrange the squares?
- How can you tell whether you have to flip a shape or just rotate it to get it to match another shape?
- Besides naming the tetrominoes by their general shape (as a "T tetromino," for example), is there any way to describe them accurately so that one could be distinguished from another?

This high-interest activity can be used either as a snappy beginning or as a several-day exploration, depending on the needs and interests of the class. In either case, maximize opportunities for students to make explanations, work systematically, and look for patterns.

An interesting discussion could develop around this question: Does the number of polyominoes for a given number of squares follow a pattern? (Two squares can be arranged one way; three squares, two ways; four squares, five ways; . . . .)

Problem 2 Most students keep their figures in the plane, but some students use three dimensions in their solutions. This is very creative and praiseworthy because nothing in the rules disallows it! To keep the problem manageable as the number of squares increases, however, students may *want* to restrict themselves to the plane. This can be decided or left open.

ASSESSMENT AND HOMEWORK IDEAS.......

- The "Checkpoint" problems ask students to explain how to decide if two pentominoes are the same or different. Since this is the fundamental question in the investigation, students' written responses could be used to assess the polyominoes activity and would serve as a pre-assessment for this module as a whole.
- The "Take It Further" problems are particularly interesting and could be assigned as projects or homework. If students complete one of these questions as a project, assess with a poster and/or presentation.

POLYOMINOES *(continued)*

- Since this is an introductory investigation, most teachers will not plan any formal assessment. Students' verbal explanations and presentations do offer the opportunity to assess mathematical communication skills, however.

Teaching Notes

Investigation 2.2

Student Pages 4–11

COMPARING PICTURES

OVERVIEW

Materials:

- **tracing paper**
- **rulers**
- **protractors**
- **"Find Your Partner" pictures**

In this investigation, we make the transition from an informal discussion of *same shape* to a formal definition of congruence and use of the $\cong$ symbol. As students compare pictures of common objects, they have an opportunity to clear up misconceptions or confusions about what congruence really is, and to begin to talk about testing for congruence.

Prepare (from blackline masters) a class set of pictures for the "Find Your Partner" game. Tracing paper, rulers, and protractors might be needed when students are measuring pictures to test for congruence. This will be a short activity, probably taking less than one full class.

TEACHING THE INVESTIGATION

The "Find Your Partner" activity which begins this investigation may be used as an icebreaker, or as a way to assign students to new working partners or groups. Here is what happens:

- Each student receives one page of pictures.
- Everybody moves around the room looking for someone else who has a page with *all* of the same pictures. To be considered matching, the pictures must be congruent. (Note: The challenge here is that the pictures do not appear in the same order or in the same orientation on the matching papers.)

Encourage creativity in determining a match. In one class, students placed their pictures on the window to see if they could match them exactly.

After matching is complete, have students do the remaining problems in the Student Module with their partners. The last problem requires them to write a definition of congruence.

The focal point of the investigation is the final discussion, in which class members compare these definitions, and the class agrees on a working definition of congruence. Students should leave class with a clear concept of congruence.

Here are some possible definitions of congruence, although you may very well create others. What is most important is that you realize that the notion of "same shape" is fuzzy, and needs to be stated more precisely.

Two figures are congruent if:

- their only difference is in their placement;
- one can be flipped or turned so that it can be placed over the other to fit exactly;
- one can be obtained from the other by a combination of rotations and reflections;
- all of their identifying characteristics are the same except for location;
- all corresponding measurements agree.

What's coming up: We will see later that if two polygons have the same area, then one can be cut into pieces which, when rearranged, will exactly cover the other.

Some students arrive with a fairly substantial middle-school background in geometry, while others are encountering concepts of shape and congruence for the first time. Approach this investigation with the background of your students in mind, using as much time as you think valuable on the picture-matching activities.

In the "Find Your Partner" game, students may ask, "Are two pictures the same if one is flipped over?" Use this as an opportunity to introduce the vocabulary of reflection and rotation. Ask students to describe these transformations in more detail; for example, "The second picture was created by rotating the first one 180 degrees."

ASSESSMENT AND HOMEWORK IDEAS.......

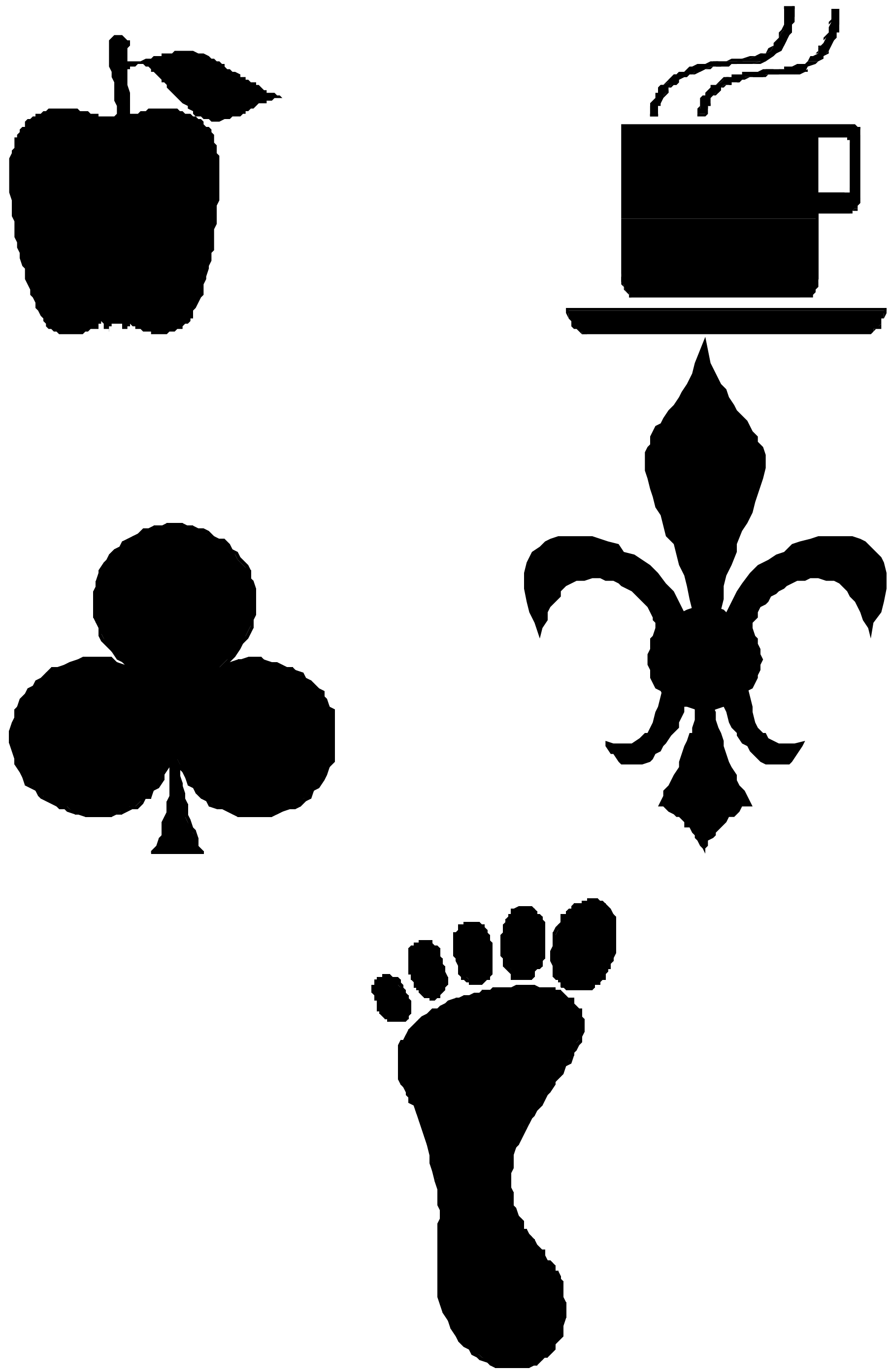
- Assign the "Write and Reflect" question at Investigation 2.3 to assess students' understanding of congruence.
- Use the "Checkpoint" (Problems 7–9) as a quick quiz or for homework.
- The "Take It Further" dictionary activity is a type of intellectual game new to many students and would make an excellent homework assignment for some classes.

Warning: The "Checkpoint" uses the term "equilateral triangle." If this is new terminology for your class, replace it with "square" for the quiz.

BLACKLINE MASTERS ....

The next several pages contain blackline masters for the "Find Your Partner" game. There are 30 pages, grouped into ten sets of three. (The first three pages contain congruent pictures, the next three match, and so on.) If you have more than 20 students, you should explain that some students may have more than one "partner."

FIND YOUR PARTNER



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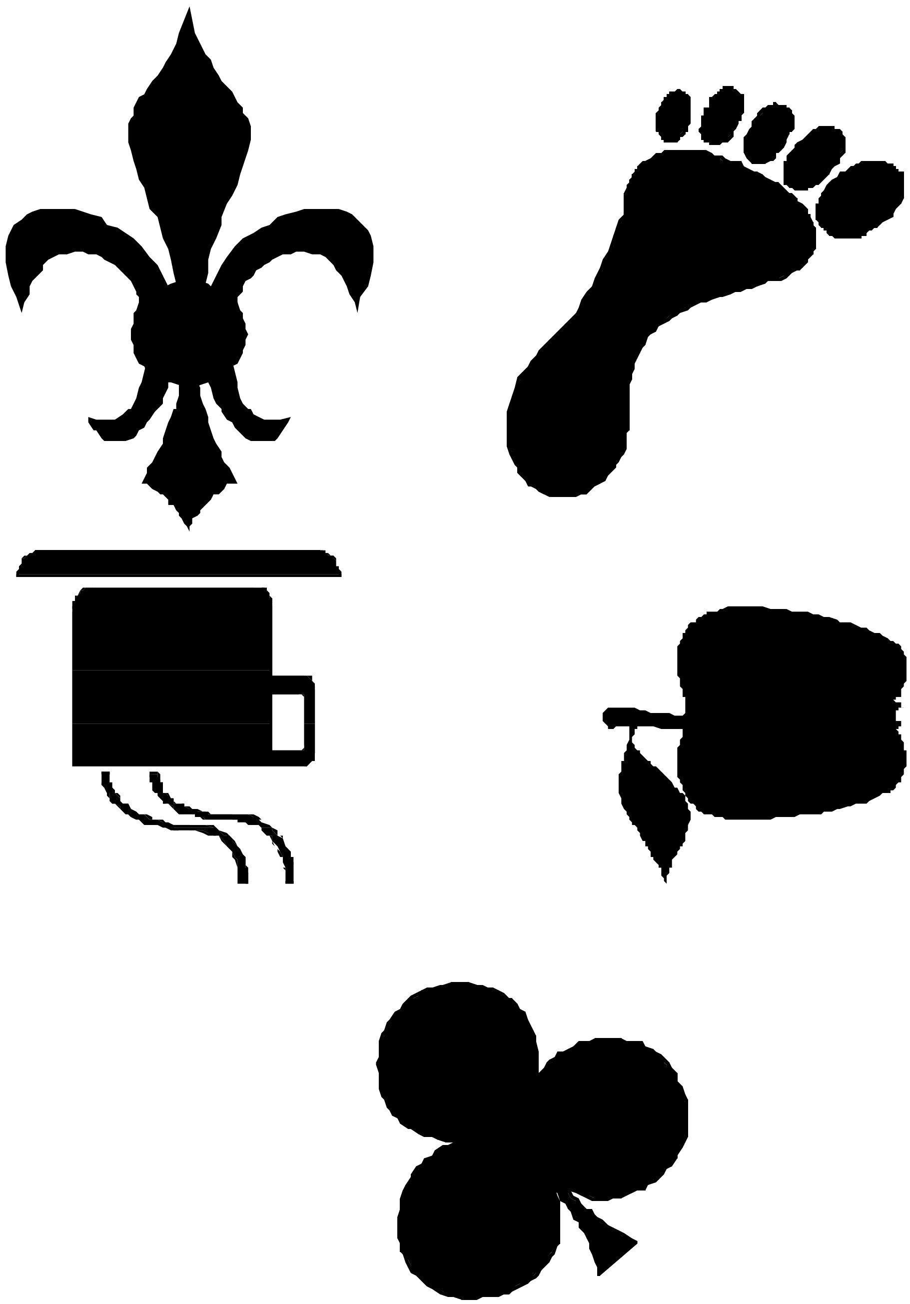

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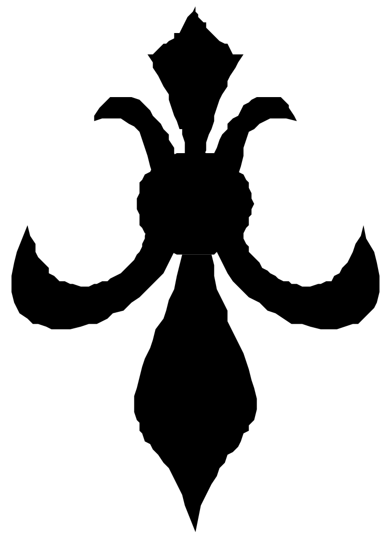

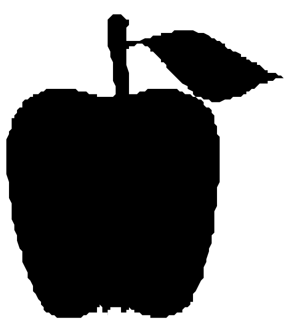

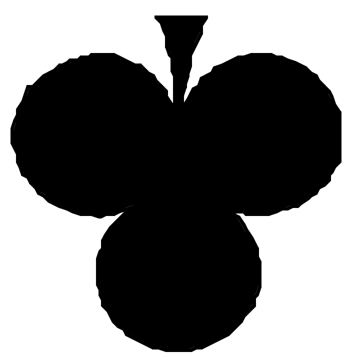

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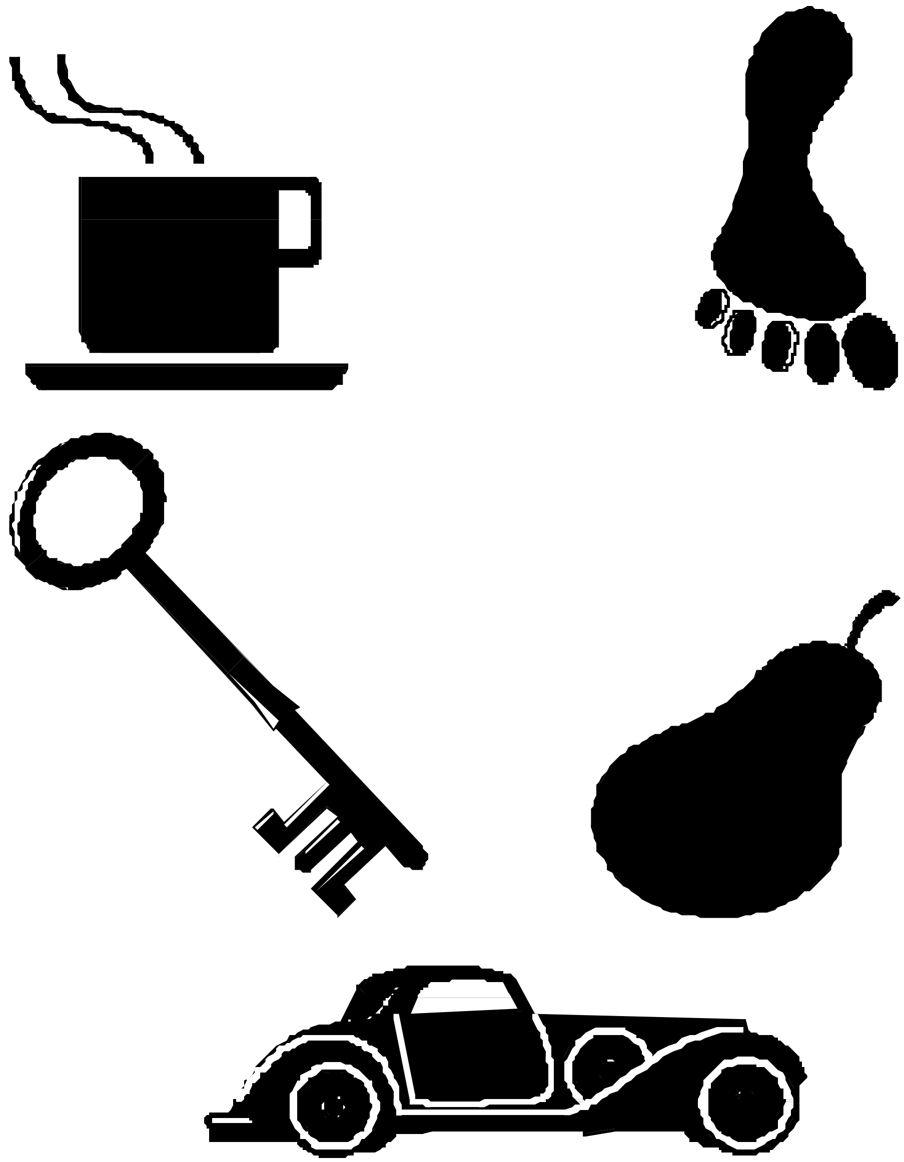

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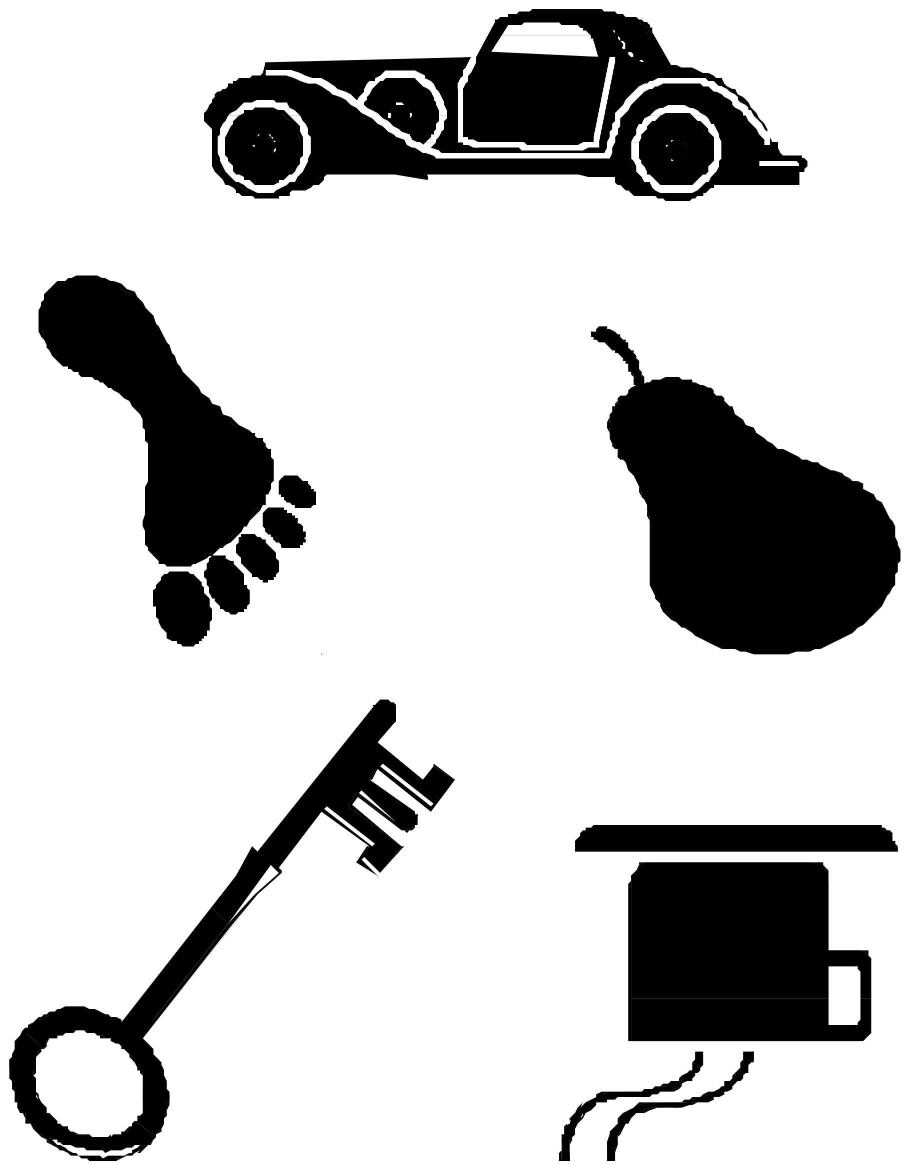

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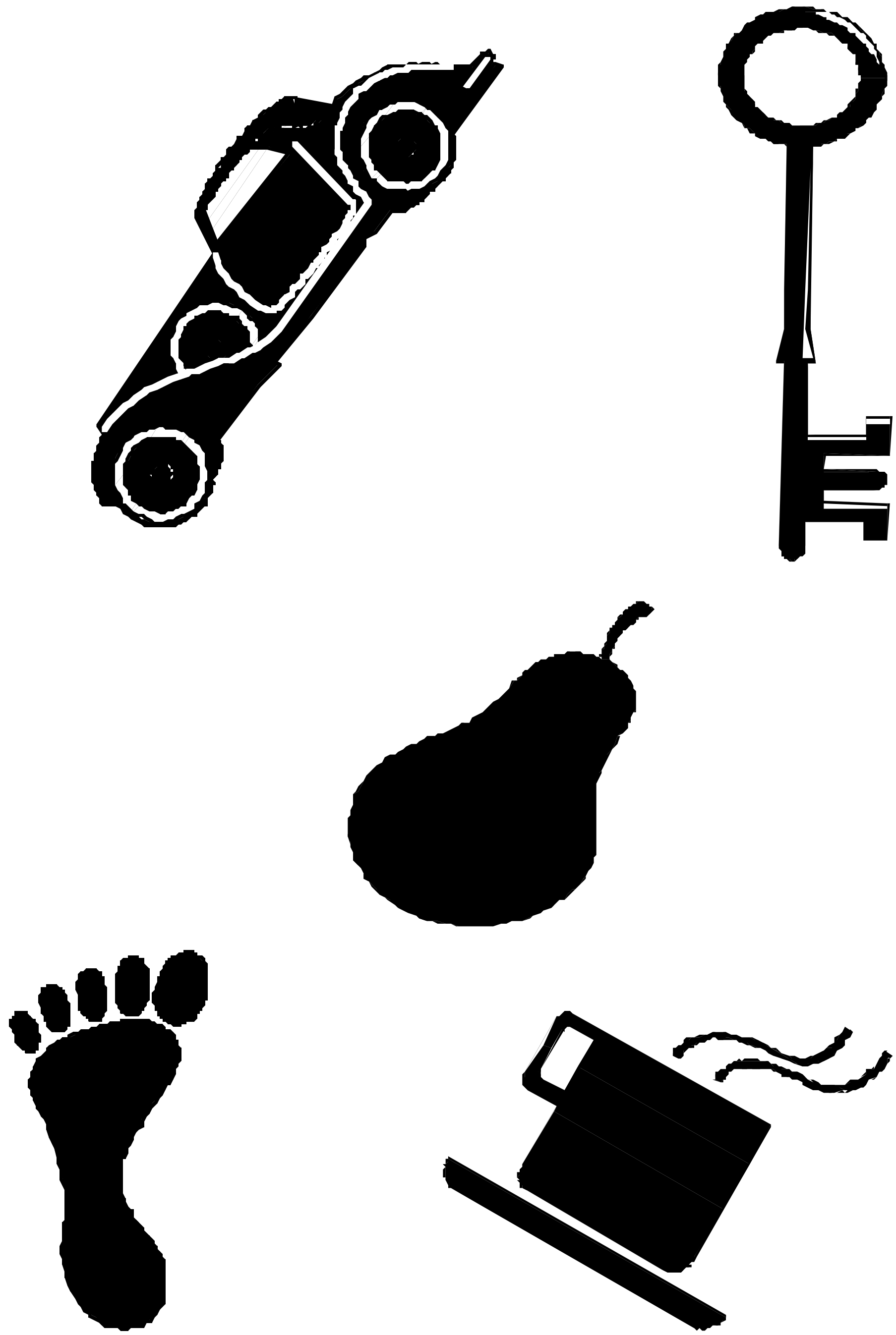

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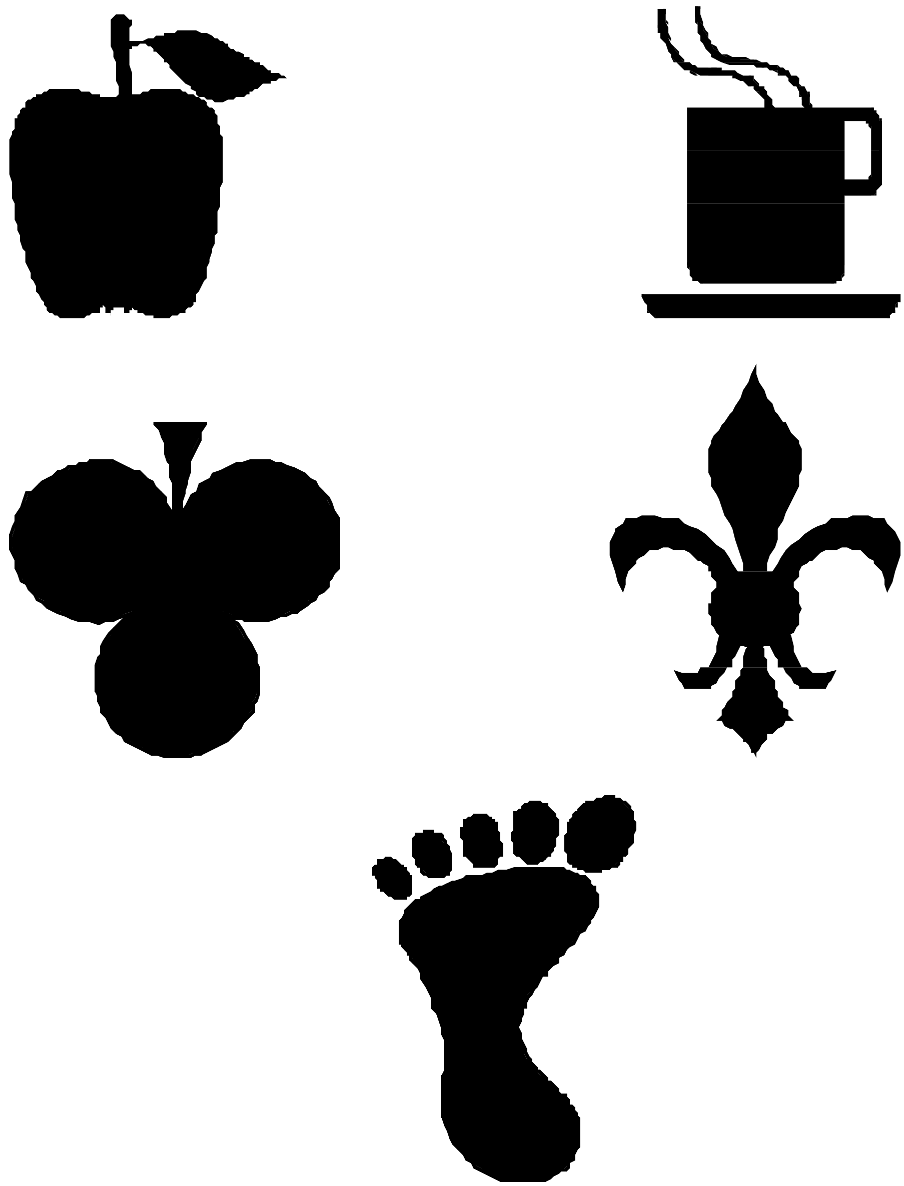

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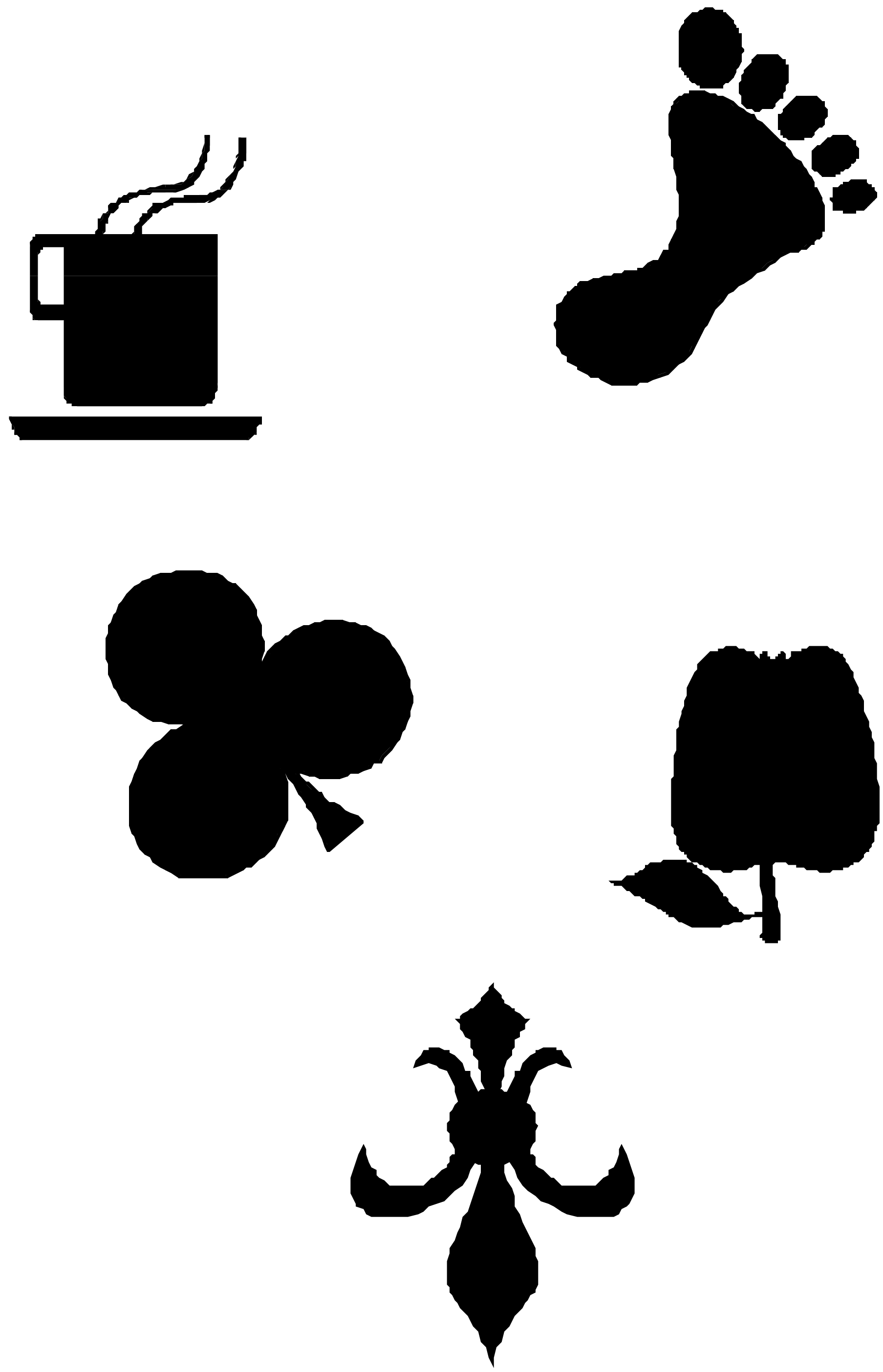

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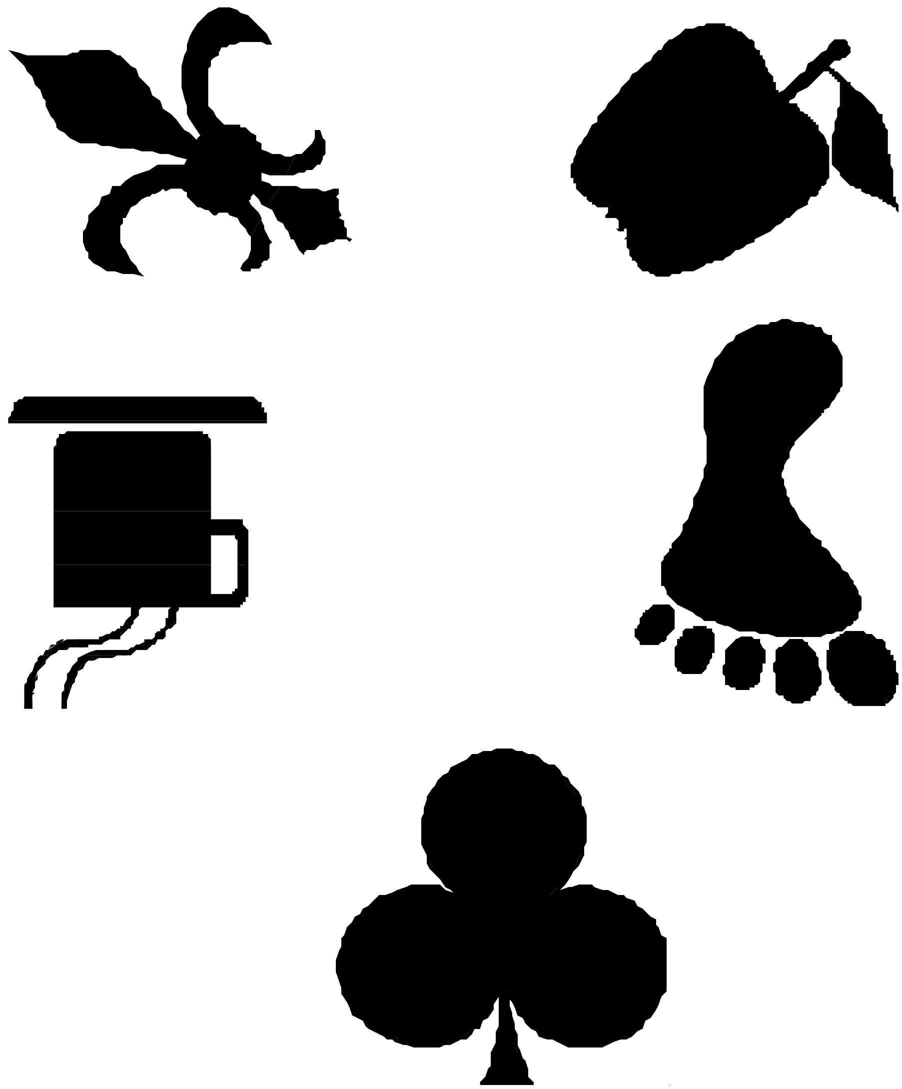


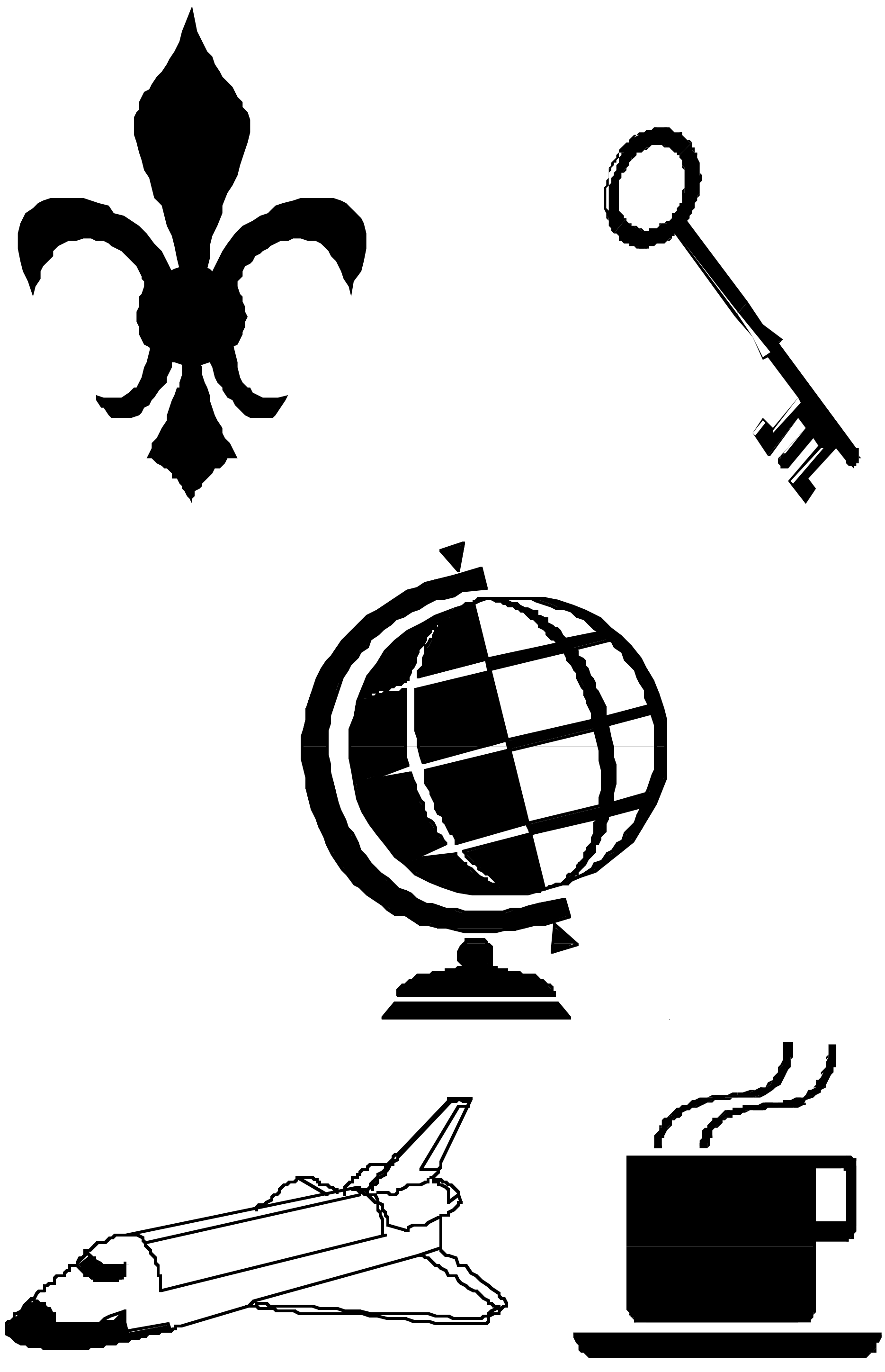
FIND YOUR PARTNER

COMPARING PICTURES *(continued)*

FIND YOUR PARTNER

FIND YOUR PARTNER

FIND YOUR PARTNER



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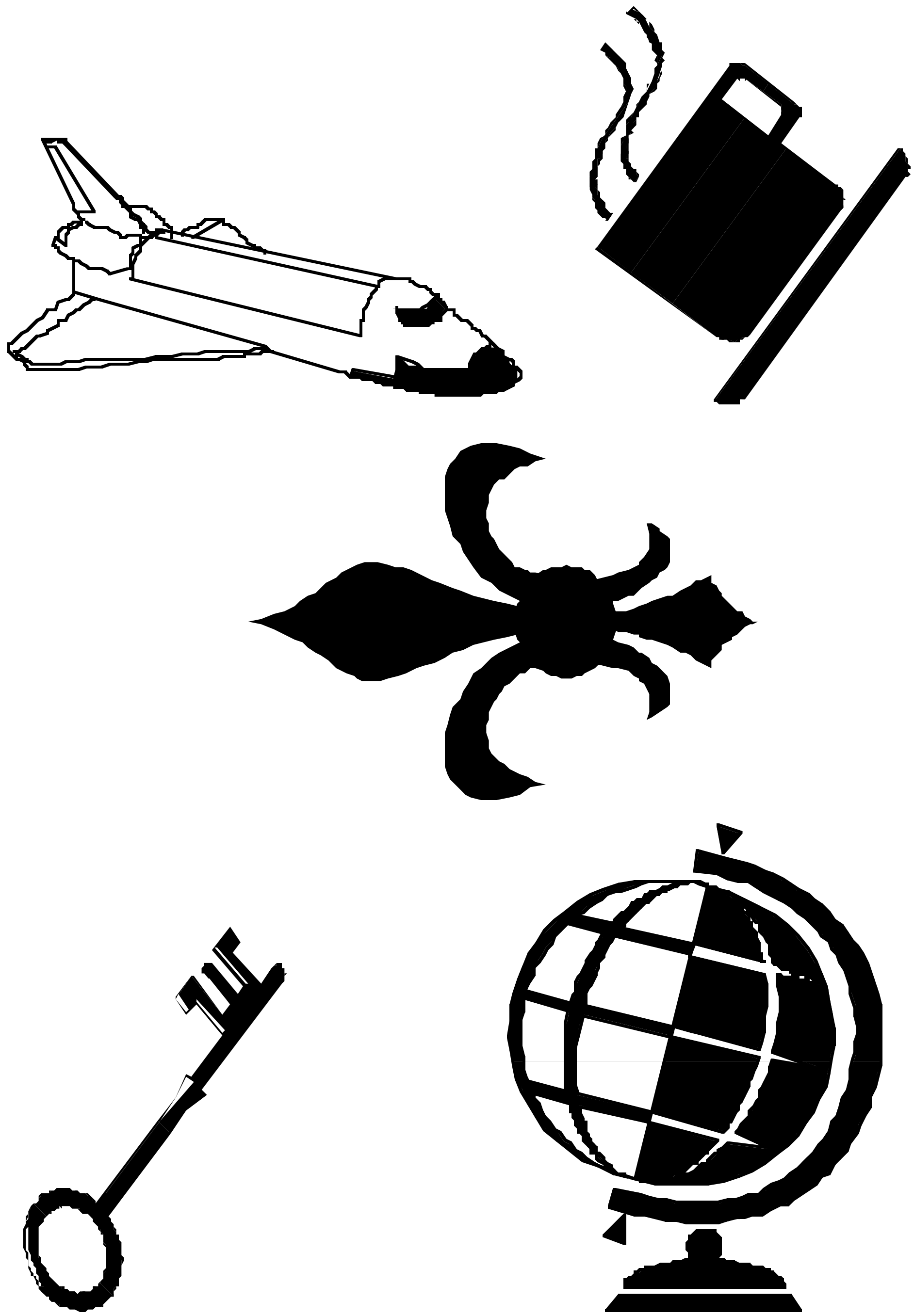

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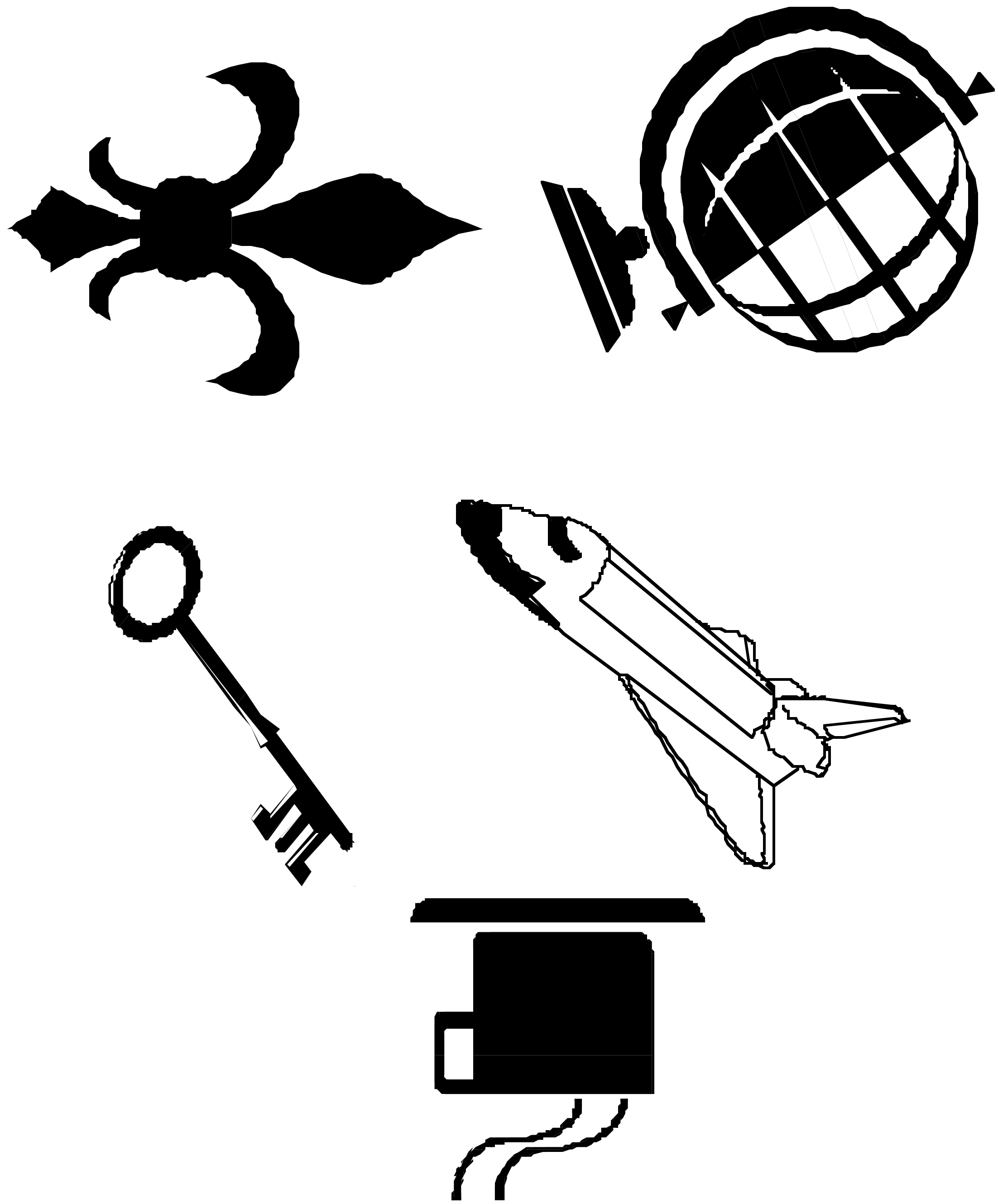

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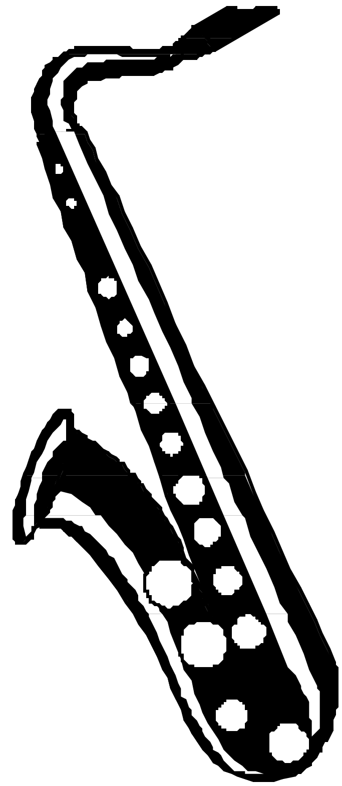

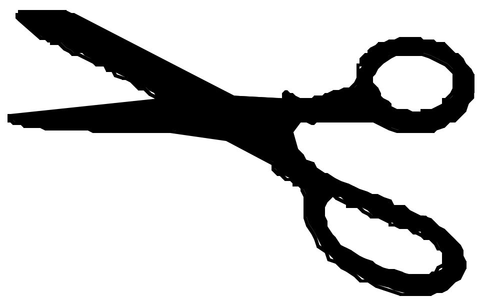

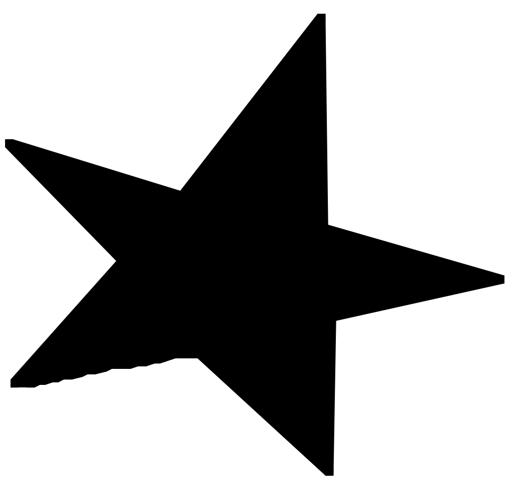

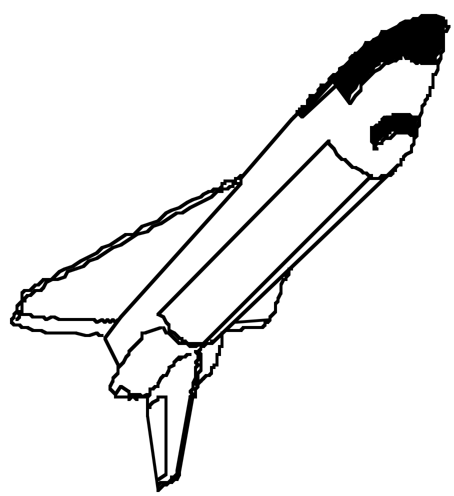

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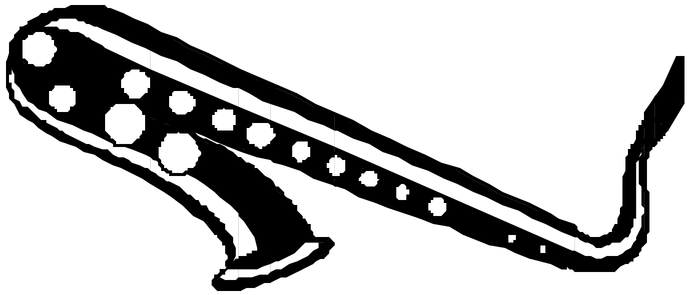

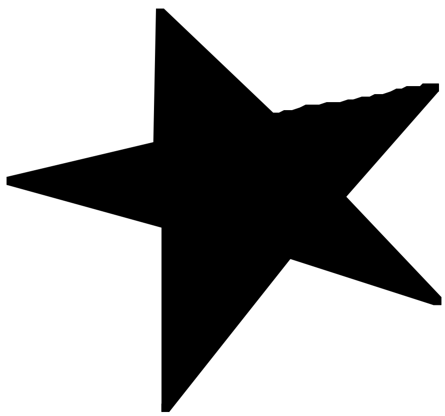

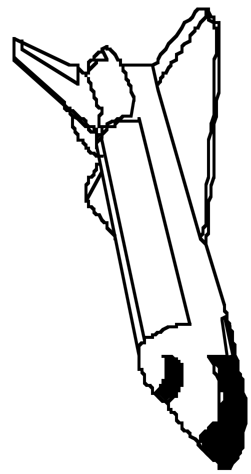

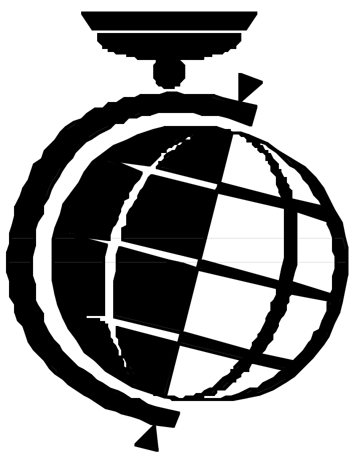

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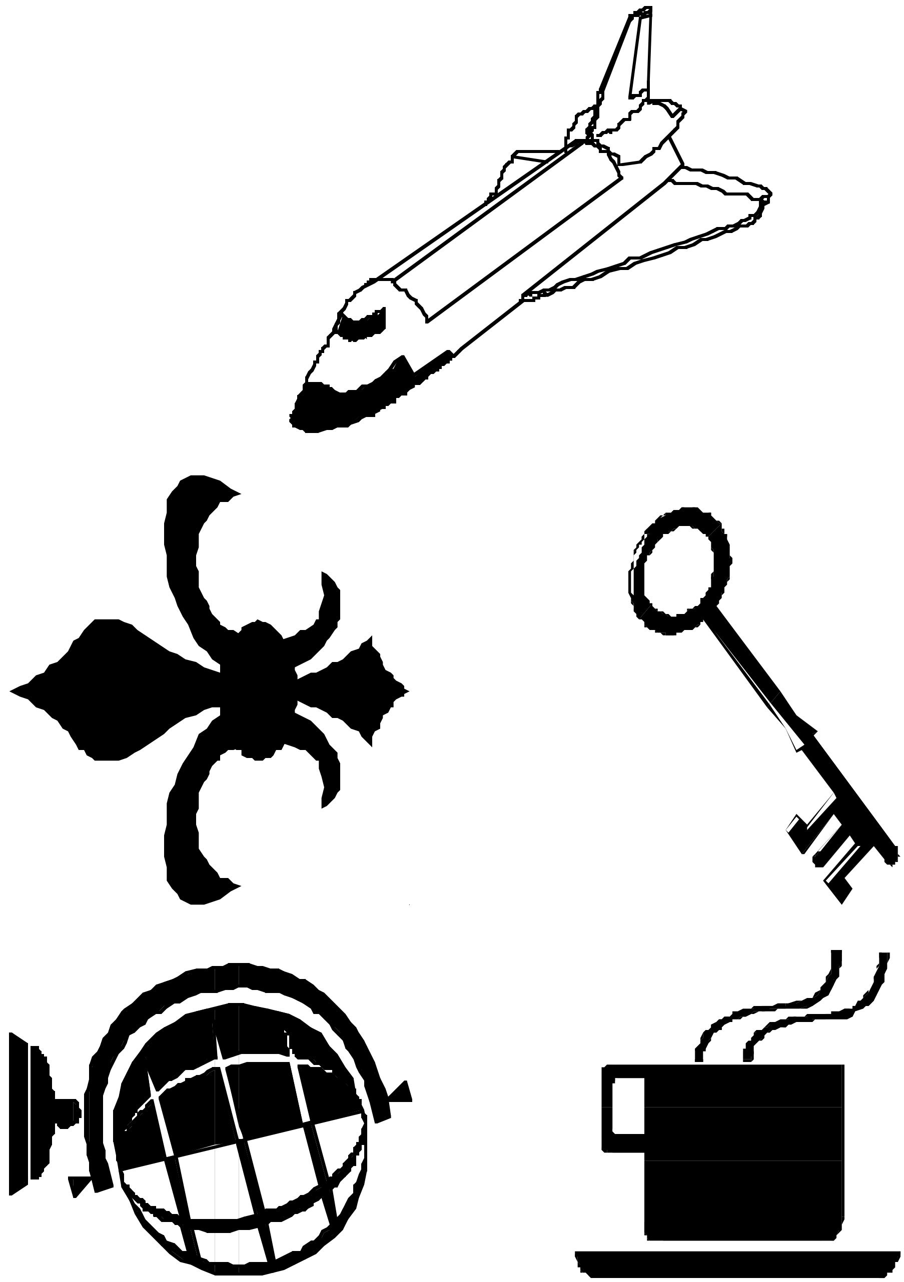

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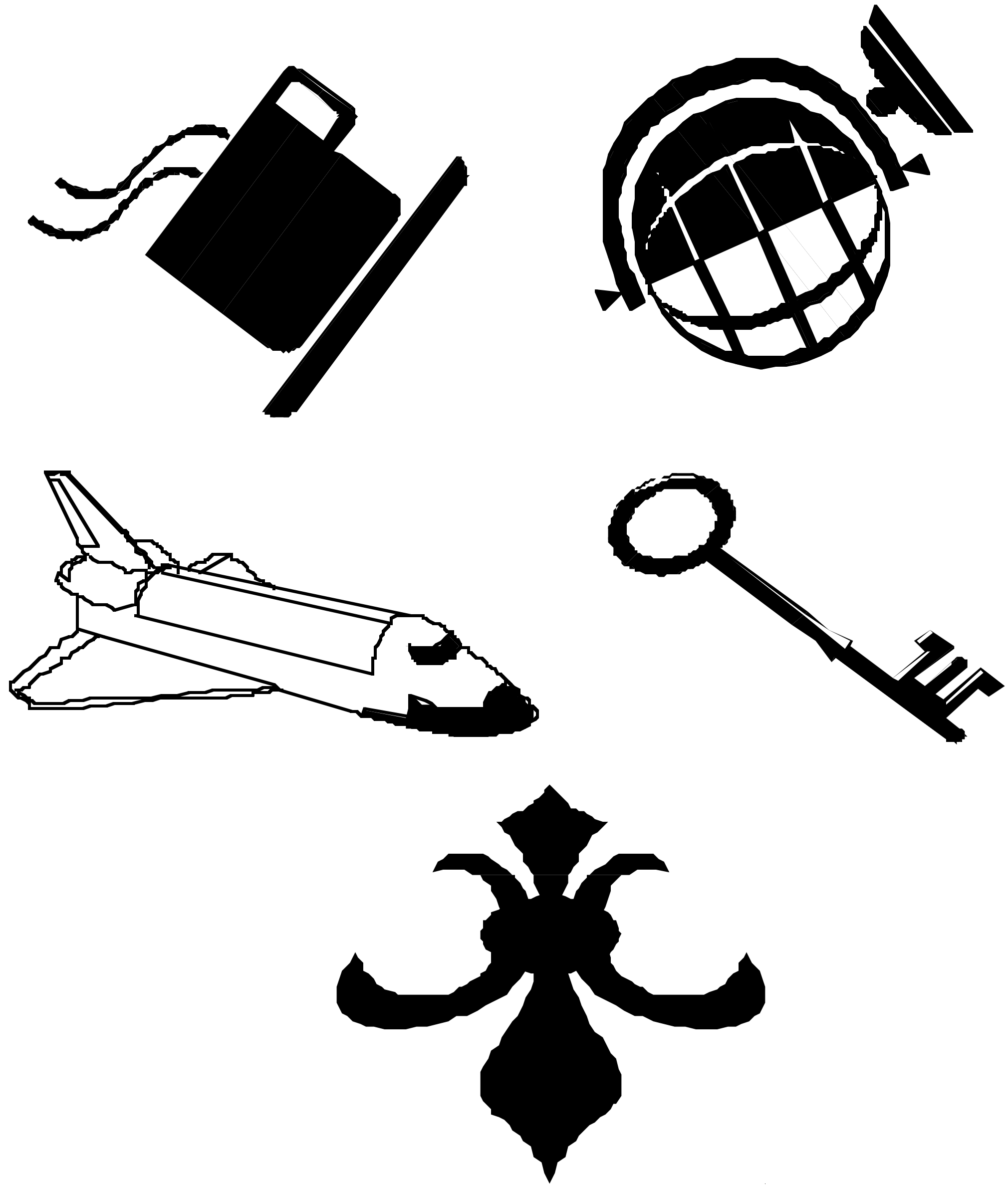

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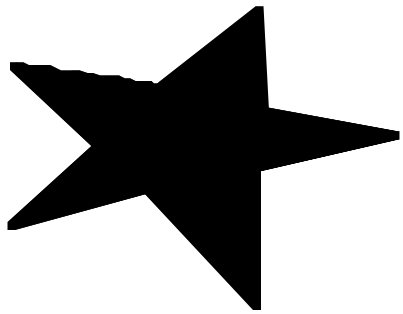

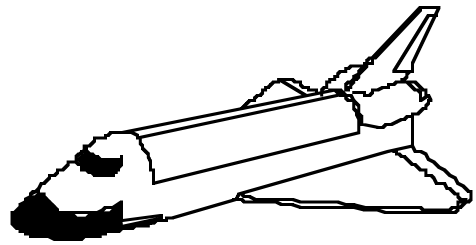

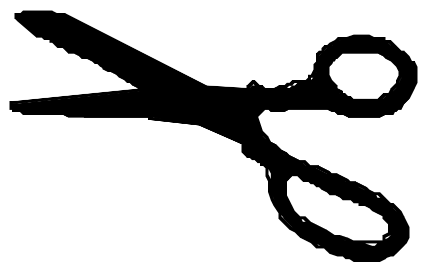

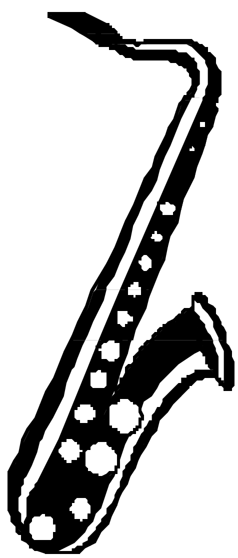

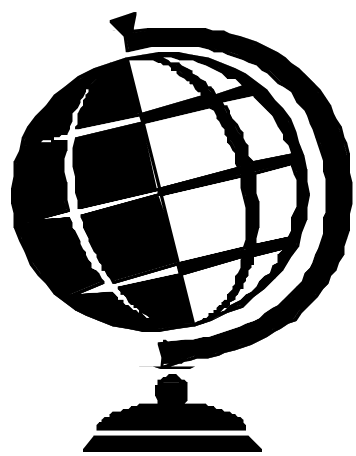

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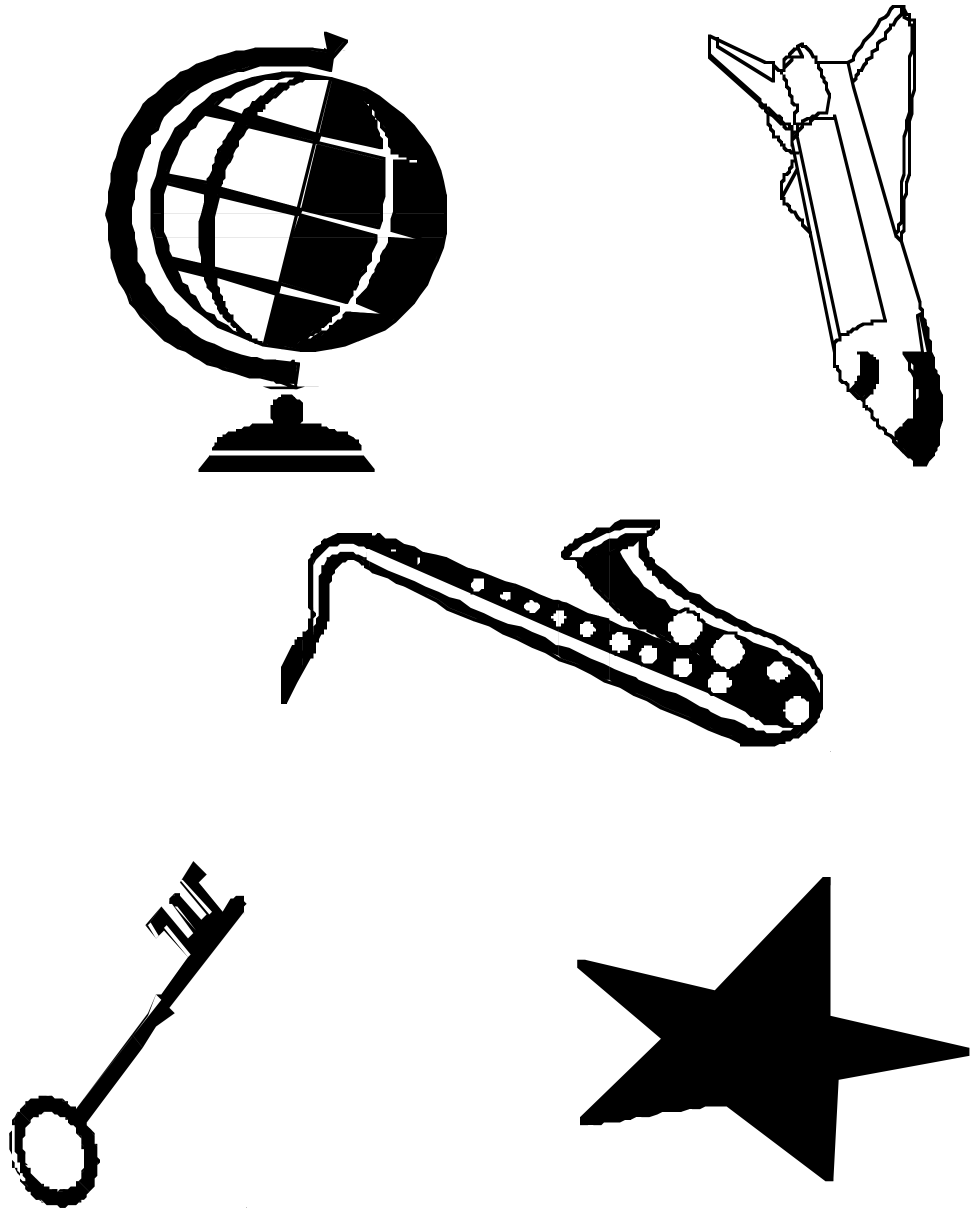

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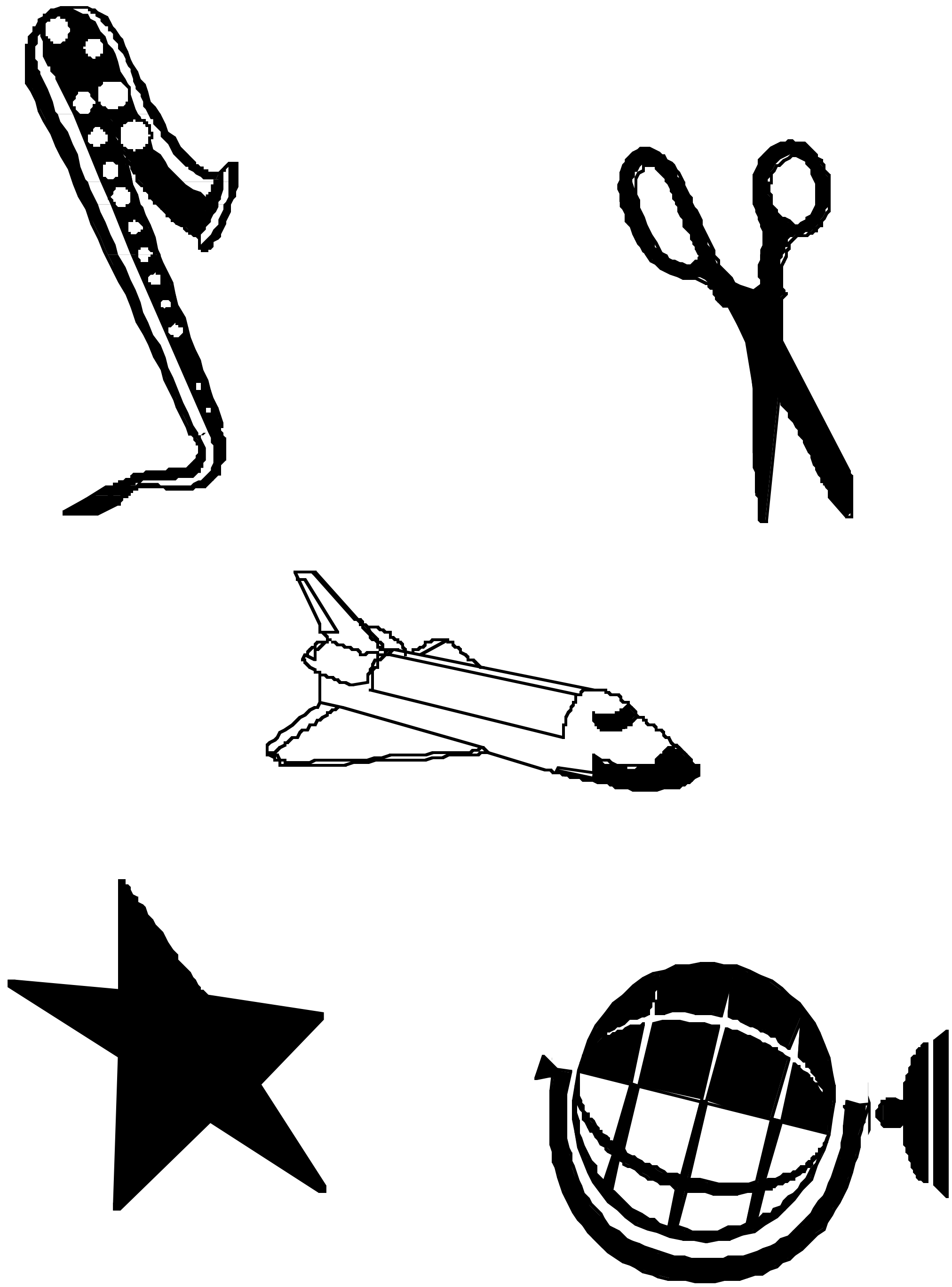

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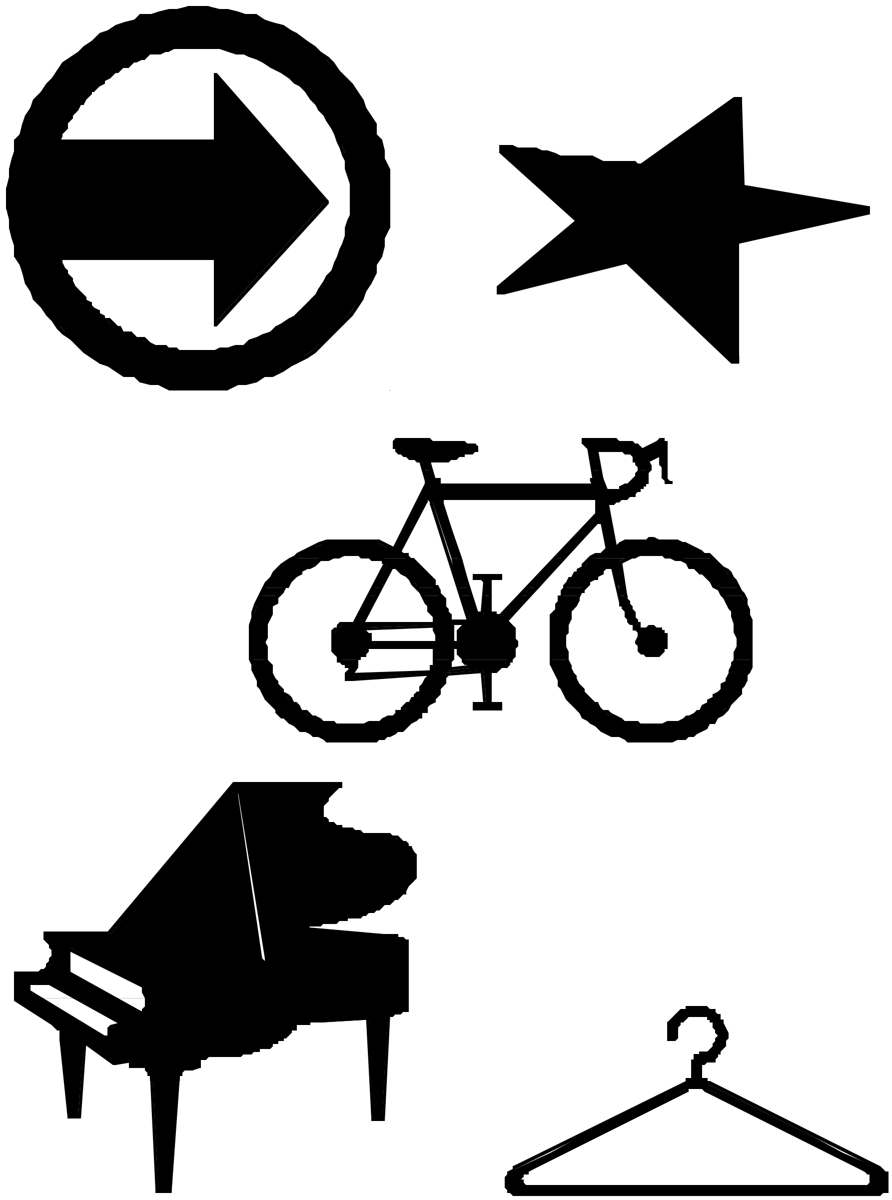

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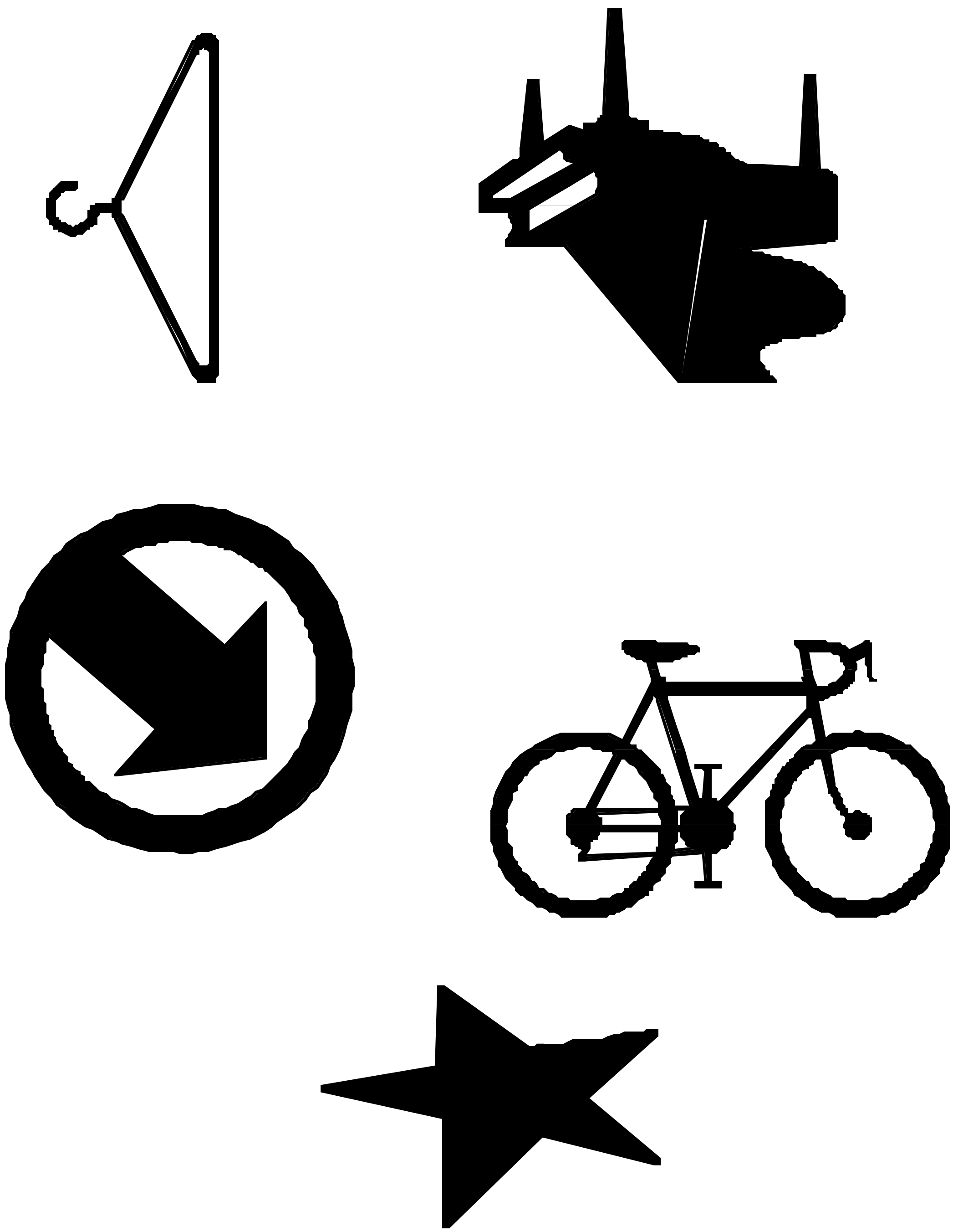

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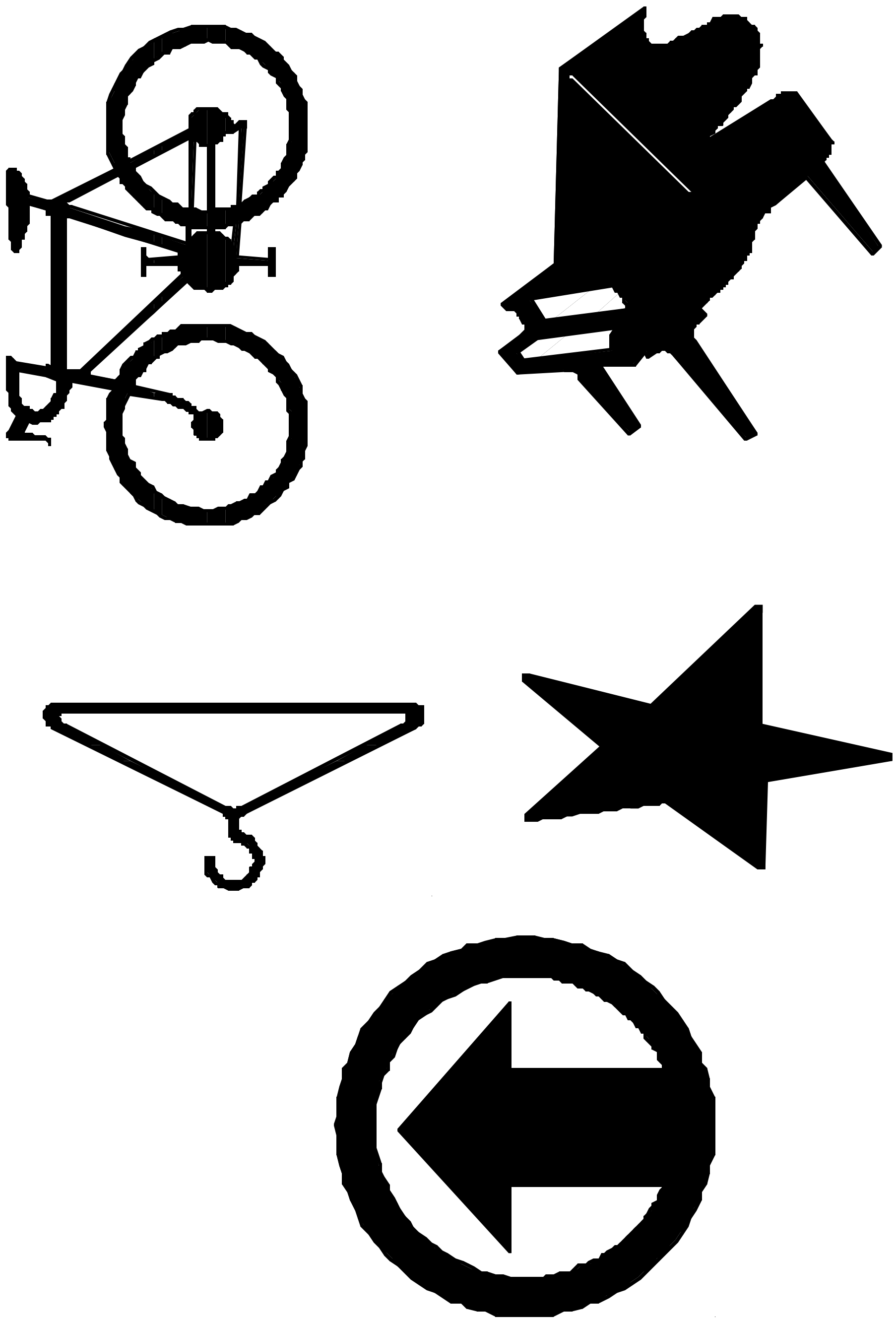

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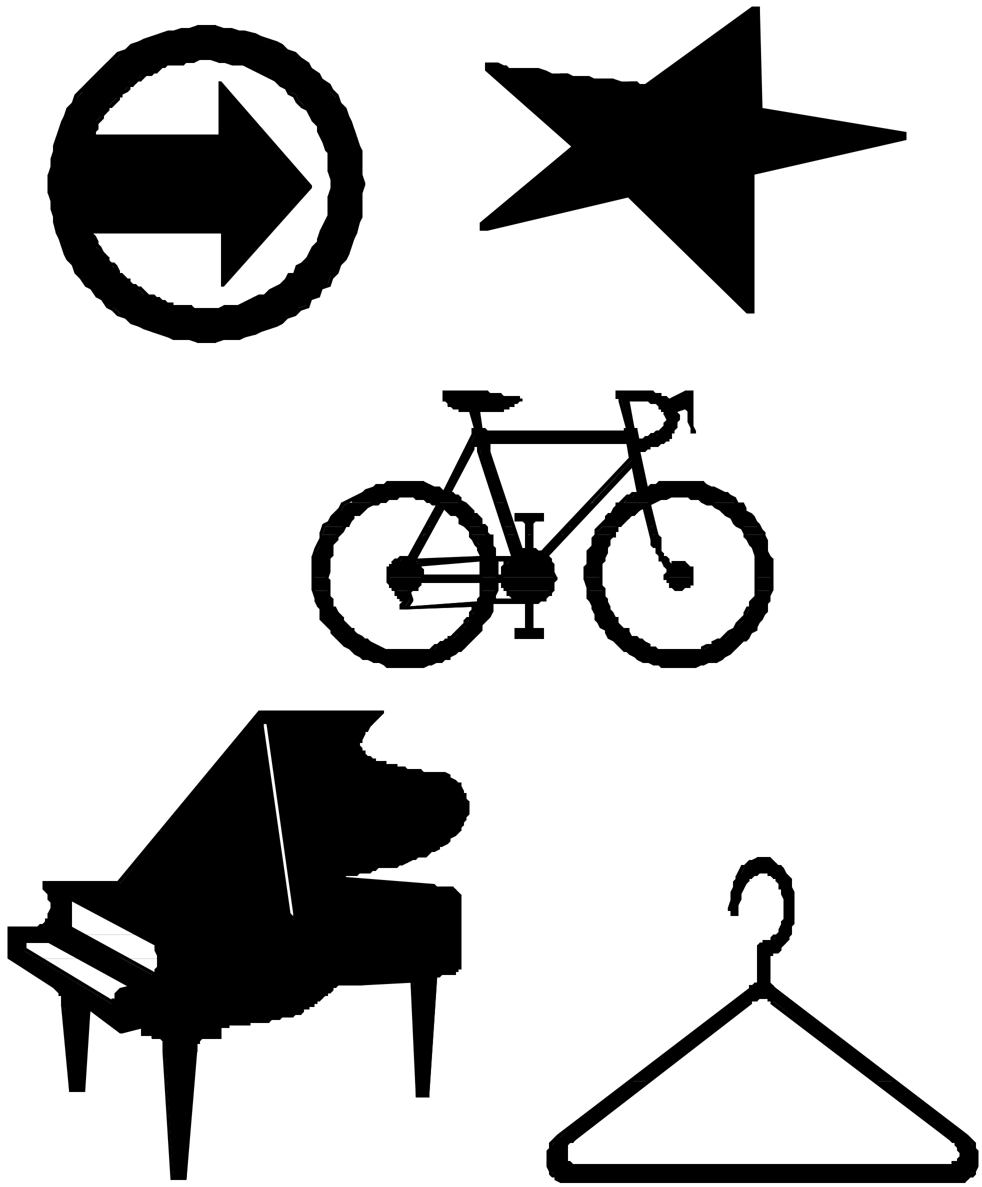

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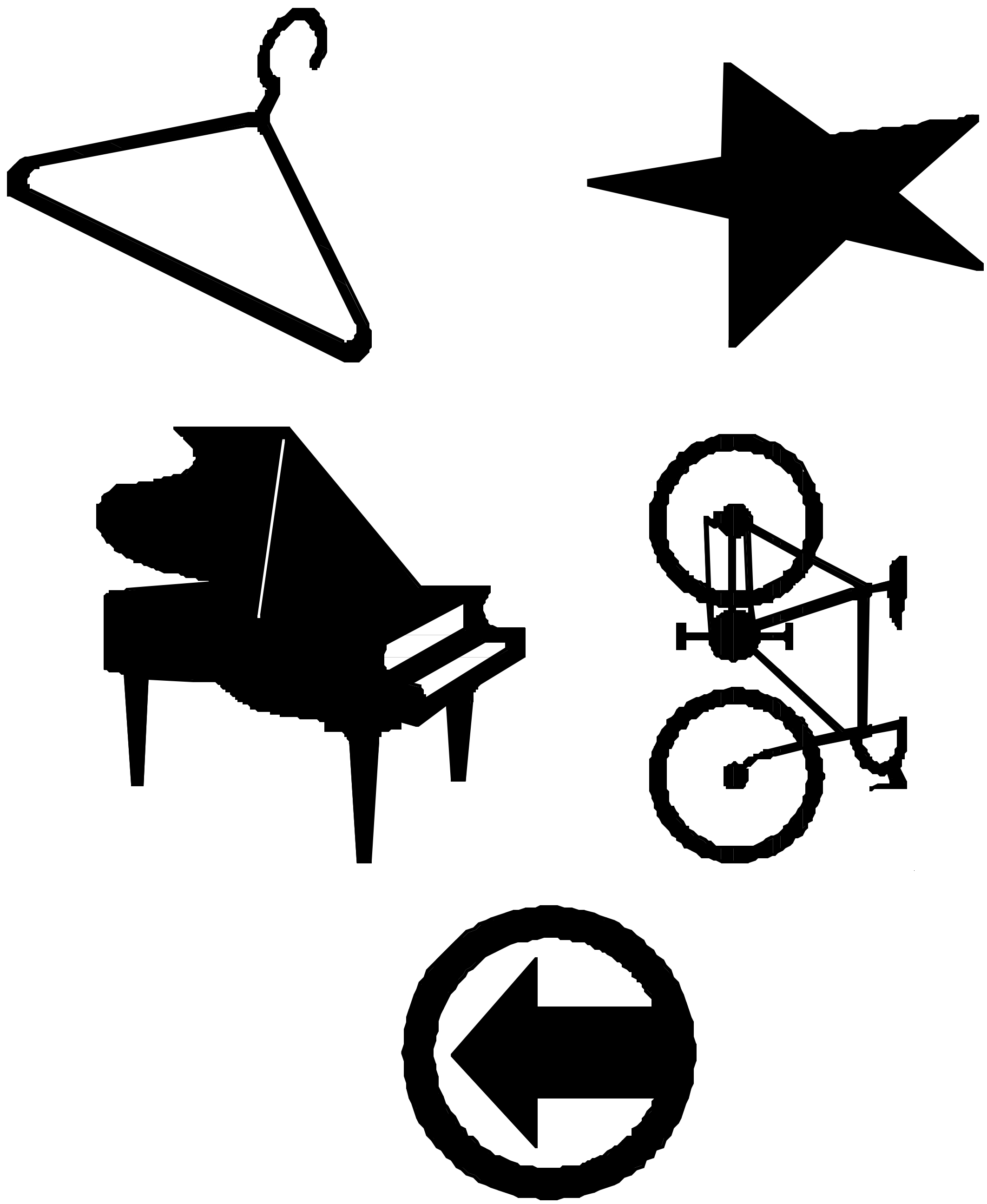

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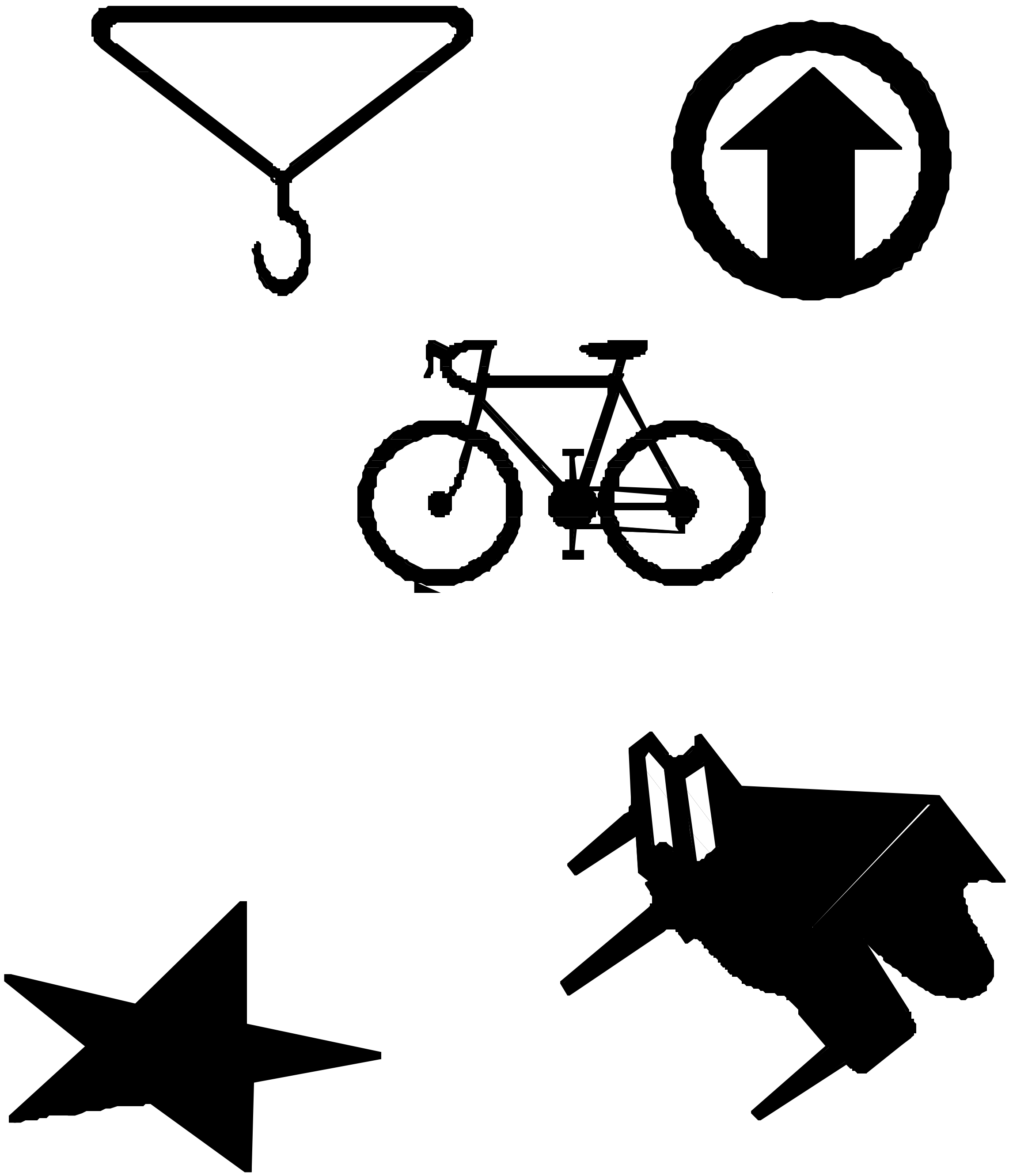


Teaching Notes

Investigation **2.3**

Student Pages 12–19

THE CONGRUENCE RELATIONSHIP

OVERVIEW

Materials:
- **rulers**
- **protractors**

In this investigation, we turn from general and informal definitions of congruence to a study of the correct mathematical notation used for communicating about congruent figures. Also, we pose these two questions:

- If two figures are congruent, must parts of the figures have equal measures?
- If two figures have certain measures that are equal, what do we know about the congruence of the figures?

Students need to have a clear concept of congruence. They will be reading, drawing, measuring, and answering questions about congruent figures. Students will need rulers and protractors, and the classroom should be organized to allow students to work individually or in groups on the problems.

TEACHING THE INVESTIGATION

This is a transition investigation. In the previous investigations, students developed a general definition of congruence. In the investigations just ahead they will start writing proofs and determining congruence for particular triangles. To move ahead successfully, students will need to accomplish these objectives:

1. to understand that measurement equality is required for congruence;
2. to understand the meaning of *corresponding parts*;
3. to read and write statements about congruent figures using correct notation.

Organize the investigation into two parts:

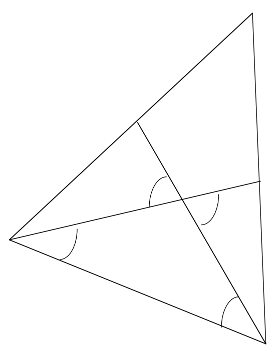
- As a full class, discuss the material in the Student Module through Problem 6. Explain terminology and keep the discussion centered on the primary lesson goals (see above).
- In small groups or working individually, students should read and do all of the remaining Problems 22–23 for homework.

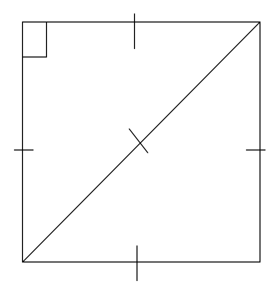
Assessment and Homework Ideas.......

At the completion of this investigation, students will have been exposed to the main ideas and language associated with the study of congruence. This would be a good time to assess their understanding before moving on to the study of triangle congruence. Here are some topics to incorporate in a quiz or written assessment:

- definition of congruence;
- identification of figures that are congruent;
- using correct symbols and notation in statements of congruence or equality;
- naming corresponding parts of congruent figures.

Also, make a few drawings with incorrect or contradictory tick marks and ask students to find and explain the errors. Here are some examples.





STRONG LANGUAGE

OVERVIEW

This investigation is very short but not small. It is an outline for an important conversation which should occur at some point in any classroom where congruence is being studied with computers or experimentation, and not solely as part of an axiomatic system.

What students need to talk about is whether measurement alone is sufficient for proof. If we measure a thousand isosceles triangles and find that in every one the altitude is also a median, have we proved that this will be true for *all* isosceles triangles?

Here we present several examples of “strong” statements (generalizations or theorems). Paired with each is a more limited statement that could be verified by measurement of the objects involved. Students are encouraged to discuss the difference between the two kinds of statements.

TEACHING THE INVESTIGATION

We encourage you to use this brief investigation as the basis for a class discussion, a reading activity, or a writing exercise. Use it just before or just after Investigation 2.5 or at a time in your class when the conversation about this dilemma is naturally emerging.

TRIANGLE CONGRUENCE

Materials:
- rulers
- protractors
- compasses
- Envelope Game

Technology: Geometry software is optional. See "Using Technology" below.

Materials Alert: Note that preparation for the envelope game must be done in advance. Allow an hour for setting up or recruit students to help the day before.

OVERVIEW

This investigation offers a hands-on approach to learning about the the tests for triangle congruence (SSS, SAS, ASA, AAS). In previous investigations, students have become familiar with the idea that corresponding parts of congruent triangles are congruent. Here the converse of that statement is first discussed, then investigated. In the investigation, we move from testing all six pairs of corresponding parts to testing all possible combinations of three parts.

This investigation is designed as a cooperative group or pair activity and will take two days for most classes. Students play two "envelope games," during which they choose triangle parts from an envelope and construct triangles from those parts. For the activity to be successful, students should already be familiar with drawing and construction techniques. (Otherwise too much time will be spent making each drawing and the accuracy level will be too poor to make judgments about the triangles.)

For the first envelope game, prepare the following for each pair of students:

- paper and drawing tools, including rulers, protractors, compasses
- an envelope containing six small slips of paper. On *each* slip of paper record *one* measurement of a triangle part, either a sidelength or an angle measure. *Important*: Be sure that the parts have letter names ($\overline{AB}$, $\angle D$, and so on). Blackline masters for two different triangles are provided below, but feel free to make up your own instead.

For the second game, a notecard with three parts listed would work just as well as an envelope, and it may be easier to make a whole class set of cards.

For the second envelope game, you will need to prepare a class set of envelopes, each containing a set of three parts of a triangle. Design the contents so that some envelopes contain sets of three angles (AAA), and some contain three sides (SSS), some SAS, ASA, SSA, and AAS. Make enough to give each group one envelope per person. *Important*: Again, be sure that the parts have letter names ($\overline{AB}$, $\angle D$, and so on).

TEACHING THE INVESTIGATION

Day One: Begin with a discussion to put the envelope game into context. Here is a rough outline:

Some classes may benefit from a fuller discussion of the question: "Is the converse of a statement always true?"

- If two triangles are congruent, then their corresponding parts are congruent (an idea the students have already discussed).
- Is the converse (if corresponding parts are congruent, then the triangles are congruent) also true?

- Assuming that the converse is true, must we always test all six pairs of corresponding parts before concluding that two triangles are congruent? If all six are not necessary, how many will suffice?

Following the discussion, play the Envelope Game. Present it as a way to test how many parts of a triangle we need to have before we know the exact shape of the triangle.

Alternate game plan: Have each group repeat the game 8 or 10 times, keeping track each time of the parts they chose and how many it took to make the triangle. With this much data, the group can come to its own decision about how many are needed, and then present its findings to the whole class.

Have each group of students play the game. One student pulls out pieces from the envelope and keeps a record of which pieces have been used. The other student(s) draw(s) the emerging triangle on paper. They should stop when they are sure that they have chosen enough pieces to build only one triangle. Each group can then proceed to the questions under "How unlucky can you get?"

End the class with a full class discussion, comparing results and deciding whether there is some fixed number of pieces necessary. The point of this first game is to allow students to discover that three pieces is usually, but not always, enough.

Definition: "Good" here means the parts make only one triangle. "Bad" means more than one triangle could be made with those parts.

Day Two: Play the second Envelope Game. It follows a similar procedure, except that now each envelope contains only three parts of the triangle. The purpose of this activity is to decide *which sets* of three parts will make exactly one triangle. Give each group several envelopes. For each envelope, students must decide:

- which combination the envelope contains (SSS, SAS, and so on);
- whether that set is *good* or *bad*.

Conclude the lesson with a class discussion in which the class agrees on the final list of the *good* sets. Identify these as congruence postulates.

This investigation draws attention to what is necessary to determine *one* triangle and leaves to the discretion of the teacher the amount of discussion necessary to establish the connection to the congruence of *two* triangles. If SSS is a "good" set of clues, it will not only produce a unique triangle, but will also determine congruence for all triangles made from the same set of three sides.

If this connection is one that might be difficult for your class, consider adapting the second game in the following way: Give each pair of students a pair of envelopes with matching clues, both with the same set of three sides or both with an SAS arrangement. Have each student construct a triangle from his clues, and then have the pair of students compare the two triangles they have made. Ask students: Are the *two* triangles made from SAS really the same exact shape and therefore congruent?

A computer demonstration with geometry software shows dramatically how two triangles can be formed with side-side-angle information given. See the figure for Problem 15.

When an activity depends on drawing, it is not unusual for students to draw incorrectly, or simply not be able to visualize alternative ways of drawing. An entire class could easily conclude, on the basis of their figures, that SSA will determine only one triangle and is a triangle congruence postulate. Plan ahead for how you want to deal with this possibility.

ASSESSMENT AND HOMEWORK IDEAS.......

Numerous problems follow the envelope games in the Student Module and can be easily used for further classwork, homework, or assessment. The triangle sketches provided in the "Checkpoint" problems ask students to identify congruence in fairly simple figures. In the "Take It Further" problems, students have to look for congruence in a wider variety of settings. Other ideas:

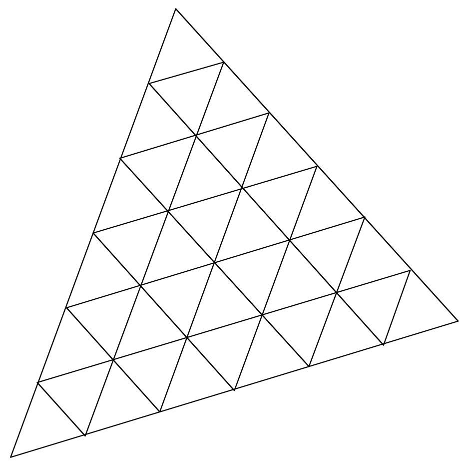
- The "Cutting Up Congruently" extension problems are fun for students to do because they seem like puzzles. At the same time, however, they will help assess whether the students know what it means for two figures to be congruent. To use them as an assessment tool, have students present their solutions to the class and justify why they think the pieces are congruent, or have students write justifications for their solutions.
- Teacher files and geometry texts are filled with problem sets which focus on identification of triangles that are congruent by SSS, SAS, ASA, and AAS. Use a page of such figures for a quiz.
- Following up on the envelope games, give an envelope quiz. Give each student an envelope with a set of clues. Ask the student to construct examples of the triangle or triangles that are possible from the parts given, and to identify the set as SSS, SAS, and so on.

USING TECHNOLOGY ....

The envelope game could be done on computers fairly easily using geometry software. Instead of drawing the triangles using hand tools, students could construct them on the computer. The constructions themselves would be a challenge, and the capabilities of the geometry software would allow students to try to stretch each constructed triangle out of shape.

MATHEMATICS CONNECTIONS

See the book *How Does One Cut a Triangle* by Alexander Soifer (published by the Center for Excellence in Mathematics Education, Colorado Springs, 1990) for more of the triangle dissections. A generalization of the Midline Theorem allows you to divide any triangle into n^2 congruent pieces by dividing each side into n pieces and connecting the points of division so that segments are parallel to the sides.



6^2 *triangles*

BLACKLINE MASTERS

Both of the sets of measures on the following page will both work well for the envelope game.

TRIANGLE CONGRUENCE (continued)

For $\triangle ABC$**:**		
$AB = 4$ cm	$BC = 9$ cm	$CA = 7$ cm
$m\angle B = 47°$	$m\angle C = 25°$	$m\angle A = 108°$

For $\triangle DEF$**:**		
$DE = 4$ cm	$EF = 6$ cm	$FD = 5.5$ cm
$m\angle E = 64°$	$m\angle F = 40°$	$m\angle D = 76°$

Teaching Notes

Investigation **2.6**

Student Pages 36–37

WARM-UPS FOR PROOF

OVERVIEW

This section of the module, Investigations 2.6–2.11, begins with a short lesson in which students construct both mathematical and nonmathematical arguments. As they move toward reading and creating more formal proofs, students learn some essential ideas:

- How you come up with a proof can be very different from its final presentation. Students learn techniques such as "visual scan" (marking up the picture and looking for ideas), "reverse list" (working backwards, deciding what you need at each stage and what you can use to get there), and flow charts.
- There are many correct ways to prove the same fact; which one you choose depends on what you know and on what you notice.
- There are also many ways to present proofs. Our materials feature two-column and paragraph proofs, as well as some presentations common in other countries.
- There is an essential difference between experimentation, in which you make and test conjectures, and deduction, in which you prove them. The only way experimentation gets you a proof is if you find a counterexample (in which case you have a proof that your conjecture is incorrect).

In working through these lessons, students should be reminded to focus on when they believe conjectures because of evidence (the triangle congruence postulates, for example), and when they have constructed *proofs*.

Investigation 2.6 presents five problems in which students are asked to "make arguments," four of them in a mathematical context. The idea is for students to move from thinking about specific cases into thinking about general proofs.

TEACHING THE INVESTIGATION

The five problems could be given for homework, with students' solutions forming the basis for the next class. Students present and discuss their solutions, talking about which arguments are convincing and why.

What's Coming Up? In the next few lessons, students will be assuming the congruence postulates in order to prove other results.

Problem 5 is particularly interesting; students are asked to assume one fact in order to prove another.

WRITING PROOFS

Materials:
- worksheet of extra proofs

Technology: Geometry software is optional. See "Using Technology" below.

OVERVIEW

This investigation begins with two very important discussions that attempt to put proofs into a full context for students.

- What are proofs? What do they look like? How do you write them?
- Why are proofs necessary in mathematics or any other context?

We begin with a discussion of how varied proofs can be, how they differ from experiments, and how they fit into the world of mathematics. A short tutorial on writing congruent triangle proofs is followed by samples of proofs written in several styles.

This investigation assumes that students understand the meaning of congruence and have been convinced that SSS, SAS, ASA, or AAS will prove triangle congruence, but have not actually written formal proofs. Have on hand a supply of congruence proofs that seem suitable for your class. Use them as examples for students to try writing in different styles.

The Student Module includes examples of proofs written by students who have studied in other countries. Before beginning this investigation, speak to your international students, and see if any of them have already learned about congruent triangles and proofs in their home country. If so, you might feature some of their work in the class.

TEACHING THE INVESTIGATION

The night before you begin, assign the first few pages and the first "Write and Reflect" question as homework.

The introductory pages of this investigation (those preceding the section labeled "A Beginner's Manual") are meant for reading, reflection, and discussion. We want students to ponder "Why prove things?" and to appreciate different styles for presenting proofs.

Begin with a class discussion about different styles of proof. Use student experts from other countries to enrich this discussion. While the primary focus of the discussion is on how many ways there are to write a proof, it is also important here to make sure that everyone understands the proof itself.

Many students are convinced of a result of this sort if a single example measures correctly.

Supplement with additional practice problems here, if needed.

Shift the class to work on the vertical angle problem. Make sure that there is a full discussion of the difference between the results you get from experimentation and those you get from deduction. Have students present their results to the whole class.

Move on to "Some Guided Practice." These problems may be done in groups as classwork or assigned as homework. Whichever approach you take, remember that these are the first proofs the students will have tried. Allow time for ideas to develop and for students to gain some confidence in writing the proofs.

Decision point for teachers: Is it important for your class to learn to write congruent triangle proofs in a particular style? The problems in the Student Module are written assuming that students will read and write in many different styles. If you prefer one style, augment these problems with additional practice in that style.

ASSESSMENT AND HOMEWORK IDEAS.......

Assessment Alert: The next two investigations include proofs with some complexity. We suggest that, before moving ahead to these investigations, teachers use a quiz or written homework assignment to ascertain whether students have an elementary grasp of congruent triangle proofs.

This investigation could easily occupy two or three days of class. In addition to completing the problems from the Student Module, suitable homework or assessment could be:

- Have students write a definition of the word "deduction." Explain what deductive reasoning is. Think of a subject in school (besides mathematics) in which deductive reasoning is used and give a specific example of how it is used in studying that subject.
- Use one of the "Write and Reflect" questions as a journal writing assignment.
- Give students a copy of a proof that is written in formal statement/reason form. Ask them to rewrite it in one or more of the other forms.
- Ask students to explain which style of proof they think is clearest or easiest to follow and why.
- Choose a set of congruent triangle proofs from another text and assign them.
- Give a traditional quiz consisting of three or four short proofs.
- Give a quiz that shows a fully-written proof which has a major missing step or circular reasoning. Ask students to explain what is wrong with the proof.
- Ask a group of students to interview a scientist and ask about how proof is used in the study of scientific facts and principles. Are major results proven using deductive reasoning?

USING TECHNOLOGY

No computer use is required, but a major focus of the investigation is the comparison of results obtained by experiment to those obtained by proof. This is a perfect place to introduce some questions for exploration on geometry software. Here are some examples:

1. Using geometry software, draw a triangle, a quadrilateral, a pentagon, a hexagon, and several other polygons of your choice. For each polygon, measure the sum of the interior angles. Record all of the sums in a chart that starts like this:

Number of Sides	Sum of Angle Measures
3	180°
4	
5	
6	

2. Does the angle sum change if the polygon changes its shape (but keeps the same number of sides)?

3. Find a pattern or rule for finding the sum of the angles if you know the number of sides in the polygon.

4. Use your rule to predict the angle sum for a decagon (10-sided polygon).

5. On the basis of your experimentation, would you conclude that this rule is always true? Explain any exceptions to the rule that you have found, or might expect to find.

For this problem, assume that the sum for a triangle is 180°, and prove your conjecture for other polygons.

6. Assume for now that your rule is, in fact, always true. Write a proof or a deductive argument that proves your rule. (For an example of a deductive argument, look in the Student Module at the arguments that prove that vertical angles are congruent.)

Teaching Notes

Investigation **2.8**

Student Pages 48–57

Analysis and Proof, Part 1

Overview

Materials:
- worksheet of extra proofs
- globe (for Problem 16)

For a beginner, doing proofs can seem like putting together a jigsaw puzzle that has no picture. If you don't know what the final picture is supposed to look like, you search randomly for tiny pieces that seem to be the right shape or color to match some other tiny piece you are looking at. Although individual pieces might lock together successfully, it is difficult to know where they fit in the bigger picture.

When you do know what the whole picture looks like, you can see each piece as a part of the whole, build bridges from section to section, place finished subsections in their approximate positions, and even frame out the whole puzzle before you fill in the interior.

This investigation attempts to give students the "whole picture" of proof: how to analyze the structure, how to fit parts together, and how to search for missing pieces in some organized fashion. To this end, three analysis techniques are presented: the visual scan, the flow chart, and the reverse list (discussed in Investigation 2.9). The techniques are explained, and a number of proofs are provided for practice.

When students begin this investigation, they should know how to write a basic congruent triangle proof using SSS, SAS, ASA, and AAS. A number of proofs are provided, but you might want to prepare an extra worksheet for your class.

Teaching the Investigation

This investigation has a traditional look. The analysis techniques are explained in the first few pages of the Student Module, and then problems are presented for practice. Teaching it can follow an equally-traditional path.

Introducing the lesson: The day before you plan to do this investigation, spend a few minutes at the end of class reading the *Car Talk* dialogue and discussing what medical diagnosis, automobile repair, and mathematical proof have in common. Assign the "Write and Reflect" question for homework. Through the discussion and writing, students should begin to *think about their thinking*. What kinds of things do they do when they analyze a problem?

Day 1: "The Visual Scan" and "Flow Chart": You can simply explain these techniques or have the class read along as students take turns reading aloud. After the explanation, it is important to write out the proof that goes with the problem just analyzed. We want students to see clearly how the analysis relates to the actual finished proof.

Note: "Take It Further" Problem 18 is a real challenge. Add it to the list for classwork if you have an eager audience!

The Student Module also contains a brief explanation of CPCTC. You might want to add in a few more proofs to allow the class to practice with CPCTC while using the Visual Scan and/or the Flow Chart. Have the class work on Problems 4–6 or use these as homework.

Days 2 and 3: Spend these days on the "Checkpoint" problems. Have the students work in small groups and then present their solutions to the class. In this problem set, we want students to analyze the proofs before actually writing them down. Consequently, the problems are fairly challenging. Read them carefully for suitability before assigning them to the class.

ASSESSMENT AND HOMEWORK IDEAS.......

- For homework, assign proofs. Use any problems from the Student Module that have not been done in class or choose a few proofs from any standard geometry text. Ask students to make a flow chart for each homework proof and show the drawing marked for a visual scan.
- A colorful alternative: Suggest that the student use a highlighter pen to color the parts of each flow chart that will make up the final proof. You will be able to "read" the proof by following the highlighted pathway through the flow chart.
- Have students look for flow charts from other settings and bring them into class to share (computer programming, debate, organizational flow charts, and so on).
- Student presentations of proofs from group work will provide good feedback about how well students understand the analysis techniques and proofs. If you want to use them for a more formal assessment, set up a check-off system for yourself ahead of time so that you can score students as they speak. Points should be given for clarity of explanation, correctness of logic, use of analysis tools, and completion of the proof.
- Quizzing students on proof (or on CPCTC) makes sense here, but any large test should be postponed until after students have completed the next investigation.
- Make up a "What's wrong here?" quiz. Give students a proof with the picture drawn and marked for a visual scan, but mark too many or too few things. Ask them to identify and correct the error.

Teaching Notes

Investigation **2.9**

Student Pages 58–64

ANALYSIS AND PROOF, PART 2

OVERVIEW

Materials:
- **Extra practice proofs**

Two extremes emerge as we teach students about proof. On one side are the few students who can look at a statement, see it as a hypothesis and conclusion, and after some thought, offer the outline of a proof. The fully-developed proof seems to appear almost magically. What goes on in the mind before the emergence of the proof is a bit of a mystery, especially to the students at the other extreme, who simply do not see how to do the proof, and who beg their teachers to return to the study of algebra and its algorithmic forms.

What *is* going on in the mind, as someone tries to come up with a proof? The thought process can be flow-chart style, discussed in the last investigation. Just as likely, we believe, is a type of backwards analysis that goes something like this:

"If I want to prove that this animal is a porcupine, then I am going to have to show that it has spines. Now where am I going to find some spines?"

We call this analysis form the "Reverse List" and treat it in this investigation as a learnable skill. Since this method begins with the question "What am I trying to prove?", this lesson begins with an explanation of hypothesis and conclusion, followed by practice in translating statements into hypothesis/conclusion form. We then describe the Reverse List method, with examples and practice problems provided.

Students should have a working knowledge of triangle congruence and some experience with proofs. The focus here is on developing analysis skills with more complex proofs. In preparation for teaching the investigation, it would be helpful to check the problems given in the Student Module. If they seem too difficult, too easy, or too few for your class, supplement with problems from any standard geometry text.

TEACHING THE INVESTIGATION

Day 1: Conduct a class discussion of the terms "hypothesis" and "conclusion." After the first two problems have been discussed, Problems 3–12 can be assigned for cooperative group work. Assign each group one of the problems to present to the class. Note that finding the hypothesis and conclusion for each statement should take only a short time, but the proofs and counterexamples may be difficult for some students.

Homework hint: Save two problems to be assigned as homework: one that has a valid conclusion, and one that is not true. Problems 5 and 6 are suitable.

Analysis and Proof, Part 2 (continued)

Day 2: Devote the class period to a full explanation of the Reverse List technique. First, examine carefully the example that is presented in the Student Module. Then analyze one or two more examples as a full class. It is important to choose proofs that will *not* be obvious to students. We want them to see how they can use this technique to actually figure out something they couldn't figure out by just looking at the proof.

The remaining problems in the Student Module will provide sufficient material for continued classwork and/or homework using the Reverse List technique.

The completion of this investigation represents a significant break point in the module as a whole. Plan to spend time on assessment.

By the end of this investigation, students will have experienced three analysis techniques. Encourage comparisons. Which works best for them? Which was easiest to learn? Is there another technique they use that is different from these?

As you work through the sample problems with the class, discuss strategies for dealing with dead-end situations (where the choices you have made aren't leading anywhere). It is important for students to see that backtracking, regrouping, and starting over are an integral part of working on proofs.

Assessment and Homework Ideas.......

Homework for this investigation will be easy to find. The analysis techniques are suitable for virtually any proofs that you want to use. Choose problems from the Student Module or from another source and assign some each night. You might also ask students to:

- Analyze a proof in more than one way (e.g., show both a flow chart and a reverse list).
- Analyze a proof using the Reverse List technique and write down the full outlined analysis. Then write the proof itself, showing how it is formed by following the outline in reverse order.
- Read one or both of the essays from Investigation 2.10. These are very thoughtful essays which would greatly enhance the quality of students' thinking about proof and might significantly deepen class discussions. Assign a follow-up journal entry responding to one of the essays.

Analysis and Proof, Part 2 (continued)

- Revisit a difficult proof from Investigation 2.8 and try working on it again using the Reverse List technique.
- Write a journal entry responding to one of the "Write and Reflect" questions.

The end of this investigation is a good time to make a major assessment of student learning. Here are some suggestions:

- Traditional individual testing on congruence and proof
- Testing that emphasizes the analysis techniques:
  1. In addition to actually writing a proof, students must show or explain how they analyzed the proof.
  2. Show the students a flow chart analysis and ask them to write a proof from the flow chart.
  3. Give a reverse list in incomplete form and ask the student to complete it.
- Let each student select a portfolio of work that demonstrates understanding of congruence. The portfolio choices could be free or specified by the teacher.
  1. Choose five proofs that you have done well and order them by level of difficulty. Explain what made them easy or difficult.
  2. Choose a selection of five problems that you think best show what you have learned in this module. Explain your choices.
- A search for "Strong Statements": As a class, return to the concept of a strong statement as presented in Investigation 2.4. Have the students go back through the Student Module and their classwork and homework looking for strong statements that have been proved. Write each one as a statement with a hypothesis and a conclusion. Once the collection has been made by the class (or individuals if you choose), the students could organize them by topic into a list or chart, or they could polish up the proofs and collect them into a booklet for the whole class.

Mathematics Connections ....

The converse is easy: if the triangle is isosceles, the bisectors of the base angles are congruent.

Problem 18, it turns out, is very difficult. During the editing of this module, one of our editors gave us the proof that appears in the Solution Resource. What follows is a proof worked out by one of the authors that uses complex numbers and trigonometry.

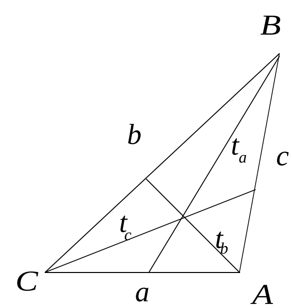
Here is the basic idea:

> Show that longer sides receive shorter angle bisectors. So, if two sides are not congruent, the angle bisectors to these sides are not congruent.

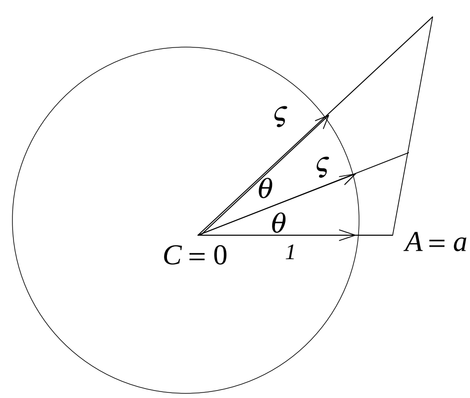
In other words, we want to express the length of an angle bisector in terms of the lengths of the sides of the triangle. Then we could work with algebra instead of geometry. We got one version of this idea to work. Here are the details.

This is somewhat unusual notation (*b* is opposite *A*, and so on), but we started like this and didn't bother to change it.

The angle bisector in terms of the sides Suppose the sidelengths of a triangle are a, b, and c. As usual, let the vertices of the triangle be A, B, and C, and let t_s be the length of the angle bisector to side s.



Let's concentrate on t_c and put the whole thing on the complex plane, with C at the origin and A along the real axis. Then let $\theta = \frac{1}{2}m\angle C$ and let $\zeta = \cos\theta + i\sin\theta$. Then $B = b\zeta^2$, $C = 0$, and $A = a$ as in the following picture:



Also, the intersection of the bisector of $\angle C$ with $\overline{AB}$ is $P = t_c\zeta$. But P is also between A and B, so $P = a + r(b\zeta^2 - a)$ for some real r. Hence we have to solve the equation

$$\begin{aligned} t_c\zeta &= a + r(b\zeta^2 - a) \\ &= a - ra + rb\zeta^2. \end{aligned}$$

If $\zeta = \cos\theta + i\sin\theta$, then $\zeta^2 = \cos 2\theta + i\sin 2\theta$ by DeMoivre's Theorem.

This is really two equations (one equating the real parts, the other equating the imaginary parts) in two unknowns (t_c and r):

$$t_c(\cos\theta + i\sin\theta) = a - ra + rb(\cos 2\theta + i\sin 2\theta).$$

Setting real = real and imaginary = imaginary, we have

$$\begin{aligned} t_c\cos\theta &= a - ra + rb\cos 2\theta \\ t_c\sin\theta &= rb\sin 2\theta = 2rb\sin\theta\cos\theta. \end{aligned}$$

We're using the double angle formulas for sine and cosine here.

The second equation implies that $t_c = 2rb\cos\theta$. Substitute this in the first equation and find that

$$\begin{aligned} 2rb\cos^2\theta &= a - ra + rb\cos 2\theta \\ &= a - ra + rb(2\cos^2\theta - 1) \\ &= a - ra + 2rb\cos^2\theta - rb. \end{aligned}$$

You know you are on the right track when things start canceling.

This implies that

$$r = \frac{a}{a+b}.$$

Since $t_c = 2rb\cos\theta$, we finally have

$$t_c = \frac{2ab\cos\theta}{a+b}.$$

Analysis and Proof, Part 2 (continued)

Check out the theorem with The Geometer's Sketchpad®. It works.

This result is interesting in itself. It also gives the length of the angle bisector in terms of the triangle's parts:

THEOREM

Using the notation in the second figure, the length of the angle bisector to side c is

$$\frac{2ab\cos\theta}{a+b},$$

where $\theta = \frac{1}{2}m\angle C$.

But what we *really* wanted was t_c as a function of the sidelengths. Well, we can do that with a bit more work: Let $\alpha = m\angle C$ so that $\alpha = 2\theta$. Then we have two facts:

Fact 1

$$\cos\theta = \sqrt{\frac{1+\cos\alpha}{2}}$$

(This is a half-angle formula from trigonometry.)

Fact 2

$$\cos\alpha = \frac{a^2+b^2-c^2}{2ab}$$

(This is the Law of Cosines.)

Similarly,

$$t_b = \frac{\sqrt{ac\left[(a+c)^2-b^2\right]}}{a+c}$$

and

$$t_a = \frac{\sqrt{bc\left[(b+c)^2-a^2\right]}}{b+c}.$$

Substituting in the theorem, we have (after a bit of simplification)

$$t_c = \frac{\sqrt{ab\left[(a+b)^2-c^2\right]}}{a+b}$$

This is what we were after. Squaring both sides, we have a nice result:

THEOREM

Using the notation above,

$$t_c{}^2 = \frac{ab\left[(a+b)^2-c^2\right]}{(a+b)^2}.$$

ANALYSIS AND PROOF, PART 2 (continued)

A simplification Suppose that we want to compare t_c and t_b, the lengths of the angle bisectors to sides b and c. Scaling, we can assume $a = 1$. Then, with this simplification, Theorem 2.9 tells us that

$$t_c{}^2 = \frac{b\left[(1+b)^2 - c^2\right]}{(1+b)^2}$$
$$t_b{}^2 = \frac{c\left[(1+c)^2 - b^2\right]}{(1+c)^2}.$$

The combining and factoring below was done with a little help from a friend—the program Mathematica®.

To compare two things, subtract them:

$$\begin{aligned} t_c{}^2 - t_b{}^2 &= \frac{b\left[(1+b)^2 - c^2\right]}{(1+b)^2} - \frac{c\left[(1+c)^2 - b^2\right]}{(1+c)^2} \\ &= \frac{(c-b)\,(1+b+c)\,\left(1+b+c+3\,b\,c+b^2\,c+b\,c^2\right)}{(1+b)^2\,(1+c)^2}. \end{aligned}$$

This does it. $t_c = t_b \Leftrightarrow$ the numerator in this fraction is zero. But, since b and c are positive, the only way for this to happen is if the factor $(c-b)$ is 0; that is,

$$t_c = t_b \Leftrightarrow c = b.$$

In English:

THEOREM

In a triangle, two angle bisectors are congruent if and only if the triangle is isosceles.

PERSPECTIVES ON PROOF

OVERVIEW . . . .

Two mathematicians, Leila Schneps and Hung-Hsi Wu, write about proof and about their own experiences learning mathematics. These meditations on the nature of mathematics are delightful, full of insight, and unlike anything most of us have ever read. They are intended to further broaden your students' ideas about proof and about themselves as mathematicans. Why is proof done? Is it necessary? How do mathematicians think about proof? How do mathematicians think about mathematics? Do mathematicians think like I do?

Although these essays are particularly helpful for students who are thinking about proof, they could be read at any time. The vocabulary and writing style of the two essays differ significantly. Read both essays before assigning them.

TEACHING THE INVESTIGATION . . . .

It is difficult to talk about using these essays as teaching tools, when they simply call out to be read and enjoyed. Nevertheless, three possibilities come to mind:

1. Use the essays to inspire discussion. Ask students to read them for homework. Suggest that they compare and contrast the two ways of thinking about proof.
   - Leila Schneps describes an inductive approach to proof. Start with facts. Live with them until they start to connect. "Your picture of the landscape slowly grows clearer and clearer as the fog seems to lift." New concepts (and proof) evolve from an understanding of the particular.
   - Hung-Hsi Wu describes proof as bringing order out of chaos, connecting immense collections of facts to give mathematics its cohesion and structure. It is satisfying because it shows why things are true. In fact, we wish to prove things because we have discovered that we cannot always believe our senses or our experimental results.
2. Use the essays as the basis for a journal writing assignment. There are many ways to do this:
   - Students could read both of the essays and then write a comparison of the two.
   - Students could read one of the essays and then write a similar sort of personal statement about their learning of mathematics or their understanding of proof.

PERSPECTIVES ON PROOF *(continued)*

- You could select an interesting quote from one of the essays and ask students to respond to it in their journals. Some possibilities are:

  a. "Mathematicians often describe mathematics like a landscape, half-hidden under a swath of mysterious cloud. The beauty and attraction of mathematics is the desire to wander about that landscape until it becomes as familiar to you as your own garden." (Schneps)

  b. "Learning to find your way around in mathematics is like visiting a foreign city with no map." (Schneps)

  c. "Almost all 'true' statements in everyday life do have exceptions." (Wu)

  d. "Proofs bring order out of the seeming chaos that hangs over the thousands and thousands of mathematical theorems." (Wu)

  e. "Learning about proofs . . . fosters a predisposition to accept the edicts of reason. From this, it is but a small step to learn to accept facts without prejudice, and therewith, to lead a life without delusions." (Wu)

3. Every student is, at some time, in positions similar to those described by *both* of the essayists: working their way out of a fog, or struggling to put together a sequence of purely-objective statements that prove a conclusion. One could use these essays as the starting point of a different kind of discussion with students. That is, what can we do to help someone along who is thinking in this manner? For example, if someone is working their way through a fog, you can:

   - ask them to clarify what they are saying by voicing your own confusion;
   - objectify what they are saying by inventing names for things;
   - think of similar situations from different contexts.

Mathematicians have invented special terminology by using agricultural terms like *sheaves*, *stalks*, *cross sections*, and *bundles*. This adds a visual construct to an essentially nonvisual concept.

INVESTIGATIONS AND DEMONSTRATIONS

OVERVIEW

This investigation contains a set of eight "big" problems which will challenge students to investigate, explain, prove, and present results to the class.

Because of the size and independent nature of the eight problems, you will find a separate set of Teaching Notes for each one. The problems are rich in opportunities for computer exploration, writing, and making and testing conjectures. Each problem brings together ideas from more than one area of mathematics. All of the investigations require students to demonstrate persistence, organization, and understanding of the main ideas of this module. Consequently, these problems work well as assessment vehicles, projects, cooperative group activities, or simply as extensions of the exploration of congruence presented up to this point in the Student Module.

This set of problems is provided as a resource for teachers. Some ways these could be used successfully:

- Spend several days on one problem, and then have each student or group of students report their results to the class with a presentation or poster. Use this as a culminating assessment activity.
- For a class that works in groups, assign a different problem to each group. Have groups present findings to the class.
- Choose to spend several weeks working through most of the problems. Emphasize the connections to topics beyond congruence.
- Spend less time on the proof sections of the module and more time on these activities.
- Form a computer lab unit that lasts one or two weeks and consists of those problems which are the most computer-related.
- Assign the problems as individual projects.

To simplify your choices, a brief summary of each problem is provided here.

Perpendicular Bisectors Students prove the theorem: "Any point on the perpendicular bisector of a line segment is equidistant from the endpoints of the line segment." Follow-up problems involve paper folding, constructions, locating the center of a circle, and finding the intersection point of the perpendicular bisectors of sides of a triangle. Constructions would work well with or without computers. This is an excellent extension of congruence ideas with connections to circle topics.

INVESTIGATIONS AND DEMONSTRATIONS
(continued)

Angles and Sides in Triangles Students prove that the number of congruent angles in a triangle corresponds to the number of congruent sides. Then they investigate inequality relationships in triangles and the hypotenuse-leg congruence theorem. Some problems here are suitable for computer exploration.

Isosceles Triangle Proofs Students study four facts about isosceles triangles and prove that any two of them will prove all the rest. This activity extends understanding of medians, altitudes, and angle bisectors and provides lots of practice with proofs. Of the six proofs, only one is difficult.

A Reflection Puzzle Two congruent triangles are placed anywhere on the plane. How many reflections does it require to "move" one of them onto the other so that they coincide? This activity assumes some knowledge of reflections and is an excellent computer exploration.

A Right Triangle Dissection Students find a method for dissecting a right triangle into n^2 congruent triangles and discuss what values of n are possible.

Connecting Midpoints on a Tetrahedron Students build a tetrahedron, connect midpoints of its edges, slice off the corners along these midlines and determine the resulting solid shape. Will its faces be congruent? The investigation extends to other solids and spirals.

Congruence from Parts If there is an SSS triangle congruence postulate, is there an MMM or AltAltAlt postulate also? (MMM stands for midline-midline-midline; AltAltAlt for altitude-altitude-altitude). Note: The altitude AltAltAlt part of this investigation is very challenging.

Making Quadrilaterals from Congruent Triangles Students cut out congruent triangles and fit them together to form quadrilaterals. Quadrilaterals are named and classified according to whether their construction required flipping any of the triangles or simply sliding and rotating them. This activity provides connections to transformational geometry and a preview of quadrilateral definition and classification.

Perpendicular Bisectors

Materials:

- paper
- cups or cans to trace
- compasses, rulers

Technology: Geometry software is optional. See "Using Technology" below.

OVERVIEW

Students begin by proving that the points on a perpendicular bisector are equidistant from the endpoints of the bisected segment. With this theorem and its converse proved, students move on to a series of construction problems involving triangles and circles.

Students with some background in proving triangles congruent, or in working with reflections, will find the proofs reasonably easy. Students without this background may still verify the theorem by logical argument, folding, or computer construction.

The problems call for some work with circles, and for "constructing" perpendicular bisectors. For these problems, traceable objects (cup, can, etc.) will be needed as well as compasses.

Serious consideration should be given to doing some or all of this activity in the computer lab. Many of the problems are easily adapted for computer investigation. Suggestions are included under "Using Technology" below.

TEACHING THE INVESTIGATION

The class session should be organized to include a balance of work time and presentation time.

The investigation centers around three main questions:

- What is a perpendicular bisector?
- How do perpendicular bisectors of the sides of triangles behave?
- What do perpendicular bisectors have to do with chords in a circle?

Class discussion will probably be more coherent and productive if the presentations are done in several separate sessions, each time considering one of the main questions.

This set of problems should elicit a great variety of explanations and proofs from students. Recognize individual methods by naming them after the student creators.

ASSESSMENT AND HOMEWORK IDEAS.......

- The investigation could be a homework problem with students presenting their proofs in class the next day.
- Problem 1 is a good assessment of the investigation; the proof is similar and uses similar concepts.
- Problems 3–6 could be done for homework.
- Problem 6 is a good assessment of students' understanding of the relationship between the investigation and circles.

USING TECHNOLOGY ....

Since the computer will speed up discovering the major results, emphasize finding other new conjectures, exploring when and why they are true, and explaining all results.

The problems dealing with perpendicular bisectors of the sides of a triangle are especially suitable for investigation in the computer lab using geometry software. The directions will need slight modification. Something like this will suffice.

1. Draw a triangle using geometry software. Construct the perpendicular bisectors of all three sides.

2. Stretch your triangle into all different kinds of triangles. Notice that sometimes a perpendicular bisector seems to pass right through a vertex of the triangle. Sometimes more than one perpendicular bisector passes through a vertex. Describe the set of triangles which have

   a. exactly one perpendicular bisector passing through a vertex;

   b. exactly two perpendicular bisectors passing through vertices;

   c. exactly three perpendicular bisectors passing through vertices.

Alternate strategy: Make this more open. Don't give students this result. Simply ask them to tell about *anything else* they have found about the perpendicular bisectors, justifying their conjectures.

3. Notice that all three perpendicular bisectors meet at a point. Does this always happen? Change the triangle and study this intersection.

4. Depending on what you have found, either prove that the perpendicular bisectors *will* always meet in one point or provide a counterexample.

5. Describe anything else that you have found about the perpendicular bisectors of the sides of the triangle, where they meet, how they divide the triangle, and so on.

Angles and Sides in Triangles

Materials:
- rulers
- protractors
- compasses

OVERVIEW . . . .

This investigation begins with a review of a familiar theorem: triangles with two congruent sides have two congruent angles. The investigation moves from this familiar ground into related, but probably new, territory. Students study the relationship between angle size and sidelength in scalene triangles, and refine their ideas about triangle congruence and a modified SSA postulate.

Students must be familiar with the triangle congruence postulates. Good hand-drawing tools are necessary for thorough investigations and presentations.

TEACHING THE INVESTIGATION . . . .

Students should work on the investigation in small groups, with discussion and sharing of results before moving on to the problems.

If all students work on all problems, you may want to allow a third day for wrap-up.

You may want to assign different "Take It Further" problems to different groups on day 2 and have them share their results. For example, one group could work on HL congruence, one group on SS**A**, and one group on SSa.

Trying to use geometry software will probably *distract* most students from the real topics of investigation here because it is fairly difficult to figure out ways to construct the figures as the invesigation requires.

ASSESSMENT AND HOMEWORK IDEAS. . . . . . .

- Students should write their solutions to the investigation and answer Problems 9, 11, and 12 for homework.
- For assessment, students can present their solutions to Problems 10, 12, 13, and 14. If groups make presentations, you can require individual write-ups to complete the assessment.

Isosceles Triangle Proofs

OVERVIEW

In this investigation, students study four statements about isosceles triangles and prove that any two of them will prove all of the rest. A similar problem involving parallelograms is included in the "Take It Further" section.

Students must be familiar with the triangle congruence postulates and have experience writing proofs based on triangle congruence.

One proof (assuming statements 2 and 4 to prove the others) uses a theorem about angle bisectors that students may not know. (See Solution Resource.) Plan ahead for how you want to deal with this.

To completely solve Problem 1, students need to know a bit about similar triangles. However, that problem could be skipped without affecting the rest of the investigation.

TEACHING THE INVESTIGATION

This investigation works very well as a group activity because the task at hand divides so neatly into manageable parts. The main investigation requires the completion of six proofs. Typically, each student in the group does one of the proofs, and they all collaborate on the difficult ones.

The directions state, "Show that, if any two of the statements are given, the other two statements can be proved." Obviously, the first thing students will have to do is figure out what all the combinations are. Some students might not realize this and will jump in to do just one proof. Clarify this before they begin.

ASSESSMENT AND HOMEWORK IDEAS

- Problem 1 is a good homework and asssessment problem. On day 2, students could share their statements and discuss which ones work and which don't.

- Problem 3 is a good assessment of the idea of using statements to prove others from the list.

A Reflection Puzzle

Materials: You may want to provide any or all of the following to your students:
- **tracing or wax paper**
- **compasses, rulers**
- **cardboard cutouts of triangles**
- **graph paper.**

Technology: Geometry software is optional. See "Using Technology" below.

OVERVIEW

Students investigate this problem:

> If two congruent triangles are placed anywhere on the plane, how many reflections does it require to "move" one of them to coincide with the other?

Students must visualize, organize their investigation, try out all sorts of shapes and positions for the triangles, and then eventually argue successfully why they think their minimum is correct.

Students must be familiar with reflections. Frustration with the activity will be lessened if students have developed some easy and accurate way to perform the individual reflections (computer tools, paper folding, cardboard cutouts, and compass constructions are all options).

Some of the "Take It Further" problems use coordinates. Skip these for now if your students aren't comfortable enough with coordinate geometry to do them. You can revisit them after students have developed some facility with coordinates.

TEACHING THE INVESTIGATION

Alternate beginning: Give the main problem as homework. Then spend the first part of the class sharing results and proceeding with the investigation.

This investigation can take some time, especially if students haven't done much work with reflections. Plan to spend time clarifying the problem and having students work in groups to come up with some conjectures. Discuss their results as a whole class. For homework, students can revise their initial conjectures based on the class discussion, and write up a summary of their solution with justification.

During the second class, you may want students to work with geometry software, studying the reflection problem further. In this environment, it should be easy to test whatever conjectures they have and create demonstrations of their solutions. The "Take It Further" problems are also well suited to computer investigation.

Complete the investigation with final presentations, and at this time stretch the discussion as far as you wish into further topics in transformational geometry. If you enjoy transformational geometry—translations, reflections, glide reflections, and rotations—this would be an opportune time to introduce these ideas with your favorite activity.

Once a clear solution has emerged, ask students to describe the characteristics of two figures which could always be mapped to one another in *two* reflections, or *one* reflection.

ASSESSMENT AND HOMEWORK IDEAS.......

- For homework, students should write up the results of their investigation and answer Problems 20, 22, and 23.
- Problems 18, 19, and 21 are ideal for in-class assessment.

USING TECHNOLOGY ....

With geometry software, once a reflection line is drawn and the triangle reflection carried out, the student can move the line around and watch the triangle image move. This takes the place of *many* separate hand drawings and moves the student much more quickly to discovering the most efficient solutions to the problem.

If students will be using the computer to investigate, suggest that they draw the two triangles separately (rather than copying and pasting) because this will give them more flexibility in the placement of the second triangle.

They will have to find a way to ensure the two triangles are congruent, of course.

Besides using a computer lab as the main location for the investigation, a single demonstration computer could be set up to allow students to share their solutions and test each other's hypotheses.

A Right Triangle Dissection

OVERVIEW

In this short investigation, students look for ways to dissect a right triangle into n^2 congruent triangles (for any integer value of n). This investigation makes a good transition from *A Perfect Match* to the next module, *The Cutting Edge*. In their solutions and explanations, students demonstrate general understanding of topics from *A Perfect Match* (congruence, midlines, quadrilaterals, and parallel lines); at the same time, the triangle dissection and discussion of area and perimeter foreshadow the cutting problems they will encounter in *The Cutting Edge*.

Because students will be expected to justify their solutions, they must come into the activity with a thorough understanding of congruence.

TEACHING THE INVESTIGATION

Before breaking into groups, have someone read the problems aloud, and make sure that everyone understands what they are supposed to do.

This investigation works well as a group activity. There will probably be a number of interesting solutions that emerge, so plan to have students use overhead transparencies or newsprint to present their work to the full class. The class discussion of why each method works or doesn't work will be very valuable.

Here is an interesting extension problem:

To find total segment length, take the perimeter of the original triangle and add to it the lengths of all the segments you inserted in order to draw the smaller triangles. *Count each segment only once!*

Study the sketch that you have made showing how to divide the right triangle. What is the "total segment length" of your picture? Find a rule that will predict total segment length if you know n and you know the original triangle's perimeter.

The relationship of the small triangles to the original triangle provides an opportunity to discuss similarity.

ASSESSMENT AND HOMEWORK IDEAS

Any of the "Take It Further" problems can be assigned for homework. The extension problem above about total segment length would also work well as homework.

INVESTIGATIONS AND DEMONSTRATIONS
(continued)

If you plan to use this investigation for assessment purposes, you may need to alter the structure of the class. For example:

- Instruct each group to write up its solution, including full justification, and collect these before the presentations begin. The group can then be graded on what it found and on how well it presents its solutions.
- Stress the importance of how students justify their solutions. This will compensate somewhat for the possibility that a number of groups may find (or overhear) the same solution.
- Encourage groups to look for more than one solution.

Connecting Midpoints on a Tetrahedron

Materials: Students will be constructing and slicing corners off tetrahedra and cubes. Prepare a supply of suitable materials such as

- **clay and straws**
- **poster board**
- **toothpicks/gumdrops.**

OVERVIEW

Visualization, tinkering, looking for patterns, and asking "What if?" are all likely to occur as students try to construct new objects by slicing the corners off a tetrahedron, a cube, and other solids.

General knowledge of polygons and the definition of a tetrahedron are the only prerequisites for this investigation.

Materials will be a major issue in preparation for this investigation. Students will want to construct models to demonstrate their solutions, and some students will need models that permit slicing in order to study the problem. Besides the suggested toothpick and gumdrop construction, clay, straws attached with paper clips or pins, posterboard, and styrofoam are all workable.

TEACHING THE INVESTIGATION

Students will probably want to work in groups for this open-ended investigation. After they have found solutions to the main problem, encourage them to think of and pursue their own extensions. For example:

- How does the original solid compare to the new one in terms of surface area and volume?
- After I have sliced off the corners of the tetrahedron to make a new solid, what happens if I repeat the procedure and slice the corners off my new solid?
- What would happen if I kept on repeating the procedure over and over?

Emphasize exploration, making and testing conjectures, and communication about results.

ASSESSMENT AND HOMEWORK IDEAS.......

- Completion of models and preparation of presentations will be the most likely homework.
- Be sure to assign the "Take It Further" problem about spirals if it hasn't been done in class. Ask students to explain what they think this problem has to do with the investigation.
- Further exploration of the Platonic solids and polyhedra would make an excellent project or serve as homework.

USING TECHNOLOGY ....

You may wish to use computer software to support either the investigation or the creation of models for presentation. Any software that creates 3D models would be suitable.

Congruence from Parts

OVERVIEW ....

In this investigation, students try to prove new triangle congruence tests. We ask, "What if the midlines of a triangle were congruent to the midlines of another triangle? Would this prove that the two triangles are congruent?" The same question is posed for altitudes, medians, and angle bisectors.

This investigation forces students to dig deeply into their knowledge of geometry to explain things which they intuitively believe to be true. Students will build the habits of mind of mixing experiment with deduction, building systematic explanations with one idea following logically from others, visualizing, and searching for invariants.

While the conclusion and explanation of congruence from midlines is within reach of most students, the altitude investigation is extremely challenging and may need to be limited, revised, or even omitted (see suggestions below).

Students must come into this investigation understanding triangle congruence, being familiar with the congruence postulates (SSS, SAS, AAS, and ASA), and knowing the definitions of median, midline, and altitude for triangles. Knowledge of the Midline Theorem and area formulas will also be helpful, but this could be developed during the investigation.

Students will be trying to construct triangles from given sets of parts (for example, from three midlines) and so will need hand and/or computer drawing tools.

TEACHING THE INVESTIGATION ....

Motivation The AltAltAlt altitude question was quietly tucked away in a problem set in the first draft versions of *Connected Geometry*. Several unsuspecting field-test teachers assigned the problem to their classes, and then found that they had launched themselves (teachers *and* students) into a week-long fascinating and frustrating investigation. Is it true? Can we construct it? Can we prove it? To a certain extent, the energy of these days came from the fact that everybody knew that nobody knew the answer. Everybody was just trying to figure it out together. Although it may require extreme self-control to avoid reading the solution, consider approaching this activity in that spirit.

Investigations and Demonstrations
(continued)

Planning the Investigation Both problems will work best if students combine construction with logical analysis. Plan time and setting to encourage this mix. The midline problem is considerably easier than the altitude investigation, both in construction and in proof. You have several options for dealing with this:

- Have the whole class work on MMM then move on to the altitude problem;
- Assign MMM to some groups, and AltAltAlt to others;
- Work on both simultaneously, with students making presentations at the moments when they have progress to report.

The construction from three altitudes is quite difficult. Allow plenty of time and recognize progress towards the solution or approximate solutions.

Construction The first step in understanding these problems comes from trying to construct triangles from the given sets of midlines or altitudes. The constructions themselves are challenging, and for some students may be the main activity. Students may use hand-construction tools or computers to build the triangle from the midlines.

Students should prepare finished drawings and/or computer demonstrations showing their construction methods.

Argument As proofs and constructions emerge, students should write up their conjectures and explanations for later presentation and grading. It will be important to actively participate in the construction of arguments. Ask students to justify their conclusions, challenge them, encourage them, give them hints, and so on.

Presentations Presentations to the class will be an ongoing part of this investigation. Student will present their construction methods first, then later have a round of “progress reports” on the proofs, and conclude with final written and/or oral presentations to the class of the completed investigations.

For students who have not already done extensive investigation of midlines and altitudes in triangles, this investigation offers an opening in that direction. As they work on the constructions and try to puzzle through the proofs, students could be encouraged to look for anything they can find that is true about the three midlines and three altitudes: points of concurrency, relationships to the sides of the triangle, and so on. Many conjectures should emerge, and these should also help students in their work on the primary investigation.

These problems are challenging enough that it would be fun to post them on a computer bulletin board for math (like the *Math Forum* at http://forum.swarthmore.edu), and share progress toward solution with students and teachers in other schools.

ASSESSMENT AND HOMEWORK IDEAS. . . . . . .

- For homework and assessment, have students write short reports, including the construction and a proof or explanation of why the student thinks congruence is proven from midlines.
- For the more difficult problems (altitudes, medians, and angle bisectors) students may write or present a "Progress Report." This would consist of drawings, conjectures, and partial explanations or justifications.
- As a challenge, assign a student (or a group of students) the task of researching the solution to Problem 33 (the angle bisector problem), reading the article cited in the Solution Resource, and, hopefully, explaining it to you or the class.
- If you spend several days on this investigation, have students write about their learning in the investigation. This is somewhat different than writing up their solutions. Here they would try to keep track of anything new they were learning and talk about how they approached the problem, what they did to make progress when they didn't know the answer, what false starts they made, and whatever they have actually concluded about the problem.

***The American Mathematical Monthly* can be found in any college library. Be warned that the solution contains some trigonometry, including double-angle formulas, and some sophisticated ideas about functions. Students will likely need help understanding the last part of the proof.**

MATHEMATICS CONNECTIONS . . . .

Here's another way to think about the altitude problem: Assume you start with given altitudes h_a, h_b, and h_c in some triangle. The question is whether or not a second triangle exists, not congruent to the known triangle, but having these same altitudes.

Begin by guessing a size for the area of the given triangle; call this guess G. The guess may be wildly inappropriate, but it gives a place to start. Working from the basic area formula for a triangle, and using this value G, you can calculate that the three sidelengths are:

$$s_1 = \frac{2G}{h_a}$$
$$s_2 = \frac{2G}{h_b}$$
$$s_3 = \frac{2G}{h_c}.$$

Starting with an initial area guess, you now have three given sides for the triangle,

and, by the SSS postulate, there is at most one triangle with sides with these three lengths. That triangle has an actual area associated with it, which may be found, for example, using Heron's formula or vectors. Let this actual area be A.

Of course, in general, G and A will be different numbers, indicating that the triangle you have produced is not the original, nor even congruent to it. You want to choose a second guess for G. The ratios in the area formulas will come into play, and you will discover that any triangle with the given altitudes will have an area that is a multiple of G; so, as a second guess, choose kG, where k is some unspecified area. This time you will discover that the sides associated to a triangle of area kG, with the given altitudes, have lengths

$$s_1 = \frac{k2G}{h_a}$$
$$s_2 = \frac{k2G}{h_b}$$
$$s_3 = \frac{k2G}{h_c}.$$

All three sides are multiplied by the factor k, and this new triangle will have area k^2A. This means that any other triangle with the same altitudes as the original is a scaled copy of the original. Since congruent triangles must have the same area, you will only find another triangle congruent to the first with the same altitudes when

$$kG = k^2A.$$

In other words, $k = \frac{G}{A}$. Using this value for k, and repeating the above procedure yields the one triangle which does indeed fulfill the desired conditions.

The classical formula of Heron says that the area of a triangle is

$$\frac{1}{2}\sqrt{(a+b+c)(a+b-c)(b+c-a)(a+c-b)},$$

where a, b, and c are the sidelengths.

A group of teachers discovered that the area is also given by

$$\frac{1}{3}\sqrt{(m+n+p)(m+n-p)(m+n-p)(m+n-p)},$$

where m, n, and p are the lengths of the medians. We've never seen this in print anywhere, but the authors think it's true, and you might enjoy trying to prove it.

Making Quadrilaterals from Congruent Triangles

OVERVIEW

Materials: Set of triangles for each group, or the materials for students to make them

In this investigation, students investigate the quadrilateral shapes that can be created by joining together pairs of congruent triangles. Distinctions are drawn between those quadrilaterals that require "flipping over" one of the triangles, and those that only require sliding or rotating the triangles.

Students should be familiar with the types and properties of quadrilaterals and triangles.

Important Each set of congruent triangles should be cut out or colored so that, when they all have the red side showing, they all have the same orientation.

TEACHING THE INVESTIGATION

As students work on this investigation, encourage them to systematize their work and findings. What are all the ways to produce a kite? What do all of the mixed color quadrilaterals have in common? What are the limitations on the shapes created by same-color triangles? Which quadrilaterals required flipping one triangle? Not every group of students will choose the same organizing principles with which to describe their results, so it is important to have them describe and draw everything as clearly and accurately as possible.

As students come to conclusions from this investigation, it would be interesting to ask them to reverse the direction of the activity. Start with a particular quadrilateral, a rhombus for example, and ask, "If you want to make a rhombus by placing two congruent triangles together, what would have to be the characteristics of the triangles and of the placement?"

ASSESSMENT AND HOMEWORK IDEAS.......

Problems 34 and 35 can be assigned for homework if they are not completed in class.

MYSTERY FIGURES

The last three investigations of the module ask students to take their study of congruence one step further, considering questions about shapes that are not triangles (including three-dimensional shapes), and using what they know about congruence to construct "mystery figures" and prove things about them. None of the tests developed here will be formalized or used later in the same way as the triangle congruence postulates.

Geometry texts from 20 or 30 years ago were filled with problems in this written format, while more recently, texts usually provide figures for the problems.

Materials:

- **Compass and ruler or**
- **Computer geometry drawing tools**

Technology: Geometry software is optional. See "Using Technology" below.

OVERVIEW

The nine problems in Investigation 2.12 challenge the student to draw complex figures from a written description, then to make conjectures about the figure, and finally to explain or prove why the statements are true. Consistent with the philosophy of teaching habits of mind, we ask students to picture, draw, describe, and conjecture.

Plan to be very selective in your choice of problems; they vary greatly in level of difficulty. Checking the Solution Resource and the suggestions below should help.

Students will need drawing tools *or* access to a computer, and may require assistance with some of the vocabulary in the problems. Knowledge of congruent triangles and proof is assumed; some problems require familiarity with quadrilaterals and parallel lines.

TEACHING THE INVESTIGATION

Each of these problems is a lesson in itself. To draw, interpret, and explain one problem could take students anywhere from 15 minutes to an hour or more. Planning how to use these problems should take this into account. Several teaching models make sense:

- Choose one problem. Divide the class into groups, and have students work on the problem together. Each group then shows its drawing, explains its conjectures, and justifies its conclusions or offers a proof.
- Choose one problem and spend two days on it.

  First night homework: Draw the figure, and make conjectures about it.

  Class next day: Show correct figure and discuss conjectures.

  Second night homework: Students try to prove the statement.

  Second day of class: Present and discuss proofs.

MYSTERY FIGURES (continued)

- Use the full set of problems as "problems of the week." Each week a new problem is assigned and students have a whole week to work on it. At the end of the week, a class discussion is held and students present their work.

  For this approach, the class would begin using the problems early in the investigation, where the focus of the work would be more on making the drawing, and on recognizing congruent objects in the figure. As the investigation progresses, the students should get more and more capable of doing proofs, and might even be able to go back and prove those that were too hard when they started.

- Use one or more of the problems as the basis of a computer activity. Constructing the figure on the computer would be the first challenge. Exploration of the figure on the computer would result in a whole different set of conjectures and explanations than the sort likely to appear when the problem is approached as a proof. Students would verify the problem statement, and then go on to find out more things that could be proved about the figure.

- Finally, the Mystery Figures could be used as assessment problems. Assign one to be written up and turned in for grading, or, if it makes sense for your class, use one as a challenging test question.

TEACHING THE INVESTIGATION ....

Students really like the drawing challenge here and feel quite victorious when they get it right. Making the figures provides excellent review of the objects described: altitudes, diagonals, isosceles triangles, and so on.

When the problem says "Pick any point on the base . . . ," students may choose a midpoint. If it says "any triangle," they may choose an equilateral triangle. Plan to discuss the issue of using special cases in doing proofs. What exactly have they proved if they picked an equilateral triangle instead of a scalene triangle?

ASSESSMENT AND HOMEWORK IDEAS.......

If you plan to use these problems for assessment purposes, offer students an example of what sort of written work you expect to accompany the drawing of the figure and suggest guidelines for the drawings. Here are some ideas:

- Make your drawing fairly large (at least 5 inches across).
- Use rulers and compasses to do the construction so that your result is neat and accurate. If the problem says two things are equal or one is three times the other, make them so in your drawing.
- Label all vertices as indicated in the problem.
- Make a list of important facts that are true about the figure you have drawn.
- Make another list that contains your conjectures about the figure. This list will contain things that you think are true, but that you might not be able to prove.
- Write a paragraph (or proof) explaining why the problem statement is true.

USING TECHNOLOGY ....

Some of the problems in this investigation would make good computer activities using geometry software. Prepare a separate worksheet for students to use as they work on each problem. Include:

- Directions for the construction itself. You probably don't want to tell them how to do it, but will want to state any special restrictions you have on how they do it. For example, whether the vertices have to be labeled with the same letters as the problem.
- A clear statement of what you want them to investigate.
- Requirements for writing or sketching. Do they have to draw on paper what they created on the screen? Do they have to record measurements that back up their conjectures? If a class needs a lot of structure or is new to this sort of investigation, they may need a chart to fill in with measurements, just to get them started.
- A statement of what you expect them to turn in to you after the lab, or of how you expect them to present their results.
- If you want, you can add several questions about the figure itself or about the mathematics being investigated. For example, if a problem is about a parallelogram, several questions about the parallelogram's characteristics could be included.

Teaching Notes

Investigation 2.13

Student Pages 86–94

BEYOND TRIANGLES

OVERVIEW

Materials:
- **compasses**
- **rulers**
- **protractors**
- **lots of paper**

Technology: Geometry software

It is in the study of quadrilaterals that many students come to really understand congruent triangles. In this investigation, students use constructions and concept maps to review the properties of quadrilaterals. Further discussion of congruence for parallelograms leads into a set of problems to be investigated using geometry software. Finally, students develop their own congruence tests for quadrilaterals and other polygons. The emphasis throughout is on active investigation.

Construction tools and paper will be needed throughout. Prior experience with compass and straightedge construction would be helpful. If a computer lab is available, plan to use it for the third (and possibly fourth) day of the investigation.

TEACHING THE INVESTIGATION

Have students bring objects or photos that exemplify each type of quadrilateral to class the day you begin the investigation.

Begin with a short, full-class discussion and "show and tell" about quadrilaterals as they appear in the "real world." This should provide a low-key opportunity to review names and basic definitions.

Important: Make sure students are doing real constructions with exact measurements. Rough freehand sketches simply won't force students to think about the quadrilateral's properties.

For some classes, providing a chart to check off properties will improve the output from this part of the investigation. Across the top, place names of quadrilaterals; down the side, the properties you want them to look for.

Following the discussion, divide students into small groups or pairs and assign the set of construction problems. These drawings will serve two purposes:

1. To draw a rhombus, for example, you have to understand its main properties. So the constructions do provide a setting for reinforcing definitions.

2. Since some of the problems describe a unique figure and others describe a whole set of possible figures, students will begin to see what information is needed to fully determine a quadrilateral. This will prepare them for a later discussion on congruence tests for quadrilaterals.

While students are working on the constructions, they should be keeping track of the properties of each quadrilateral. They will need this information to do the concept map suggested in the "Write and Reflect" problem.

To pull together the final results and encourage group productivity, have groups post their drawings on the bulletin board. Then there will be five or six drawings of each problem all together, ready to compare.

Conduct a discussion which leads to general agreement on which properties are true. Then, students can begin working on Problem 11 (proving the properties). There is a huge amount of work here if all properties are proved. Divide the work in some reasonable manner, such as:

- assigning each group a small set of proofs;
- proving only selected properties;
- assigning each group one quadrilateral to work with.

Use the posted construction drawings from day one to help motivate the discussion.

After sharing of concept maps, the class should focus on ways to prove congruence for quadrilaterals. The carpenter's door-frame problem and Problems 13 through 17 could be handled in a teacher-led discussion or by having each group write and present a convincing argument for one of the problems.

Work on Problems 18–24 in the section "Applications of Analysis and Proof" (in the computer lab, if possible). These are challenging, so allow plenty of time. One problem could take a full class period. Explain to students that their investigation of the problems should have two phases: making conjectures and then proving the conjectures. After these investigations, students should do presentations (written and/or oral) explaining their conclusions and proofs.

In "Proving Congruence for Quadrilaterals," students look for congruence tests that will work for quadrilaterals. Since there are many right answers to these questions, a major benefit of doing them will come in the debate over whether an answer is correct or not. After an initial full-class discussion, assign small groups to work on Problems 27–29, which have students creating their own congruence tests. Present and debate results.

For many students, being told to "investigate" a problem in the computer lab is like being left with no directions at all. Following the needs of your class, consider providing a list of suggestions about how to investigate or a set of additional questions to answer.

Note that for Problems 1–11, not everyone defines a trapezoid to be a quadrilateral with *exactly* two sides parallel; some say that a trapezoid should have *at least* two sides parallel. This definition lets you consider parallelograms as special cases of trapezoids.

Here are some ways to prove a quadrilateral is a parallelogram:

- Show that both pairs of opposite sides are congruent.
- Show that one pair of opposite sides is congruent and parallel.
- Show that both pairs of opposite angles are congruent.
- Show the diagonals bisect each other.

For Problem 27, note that there are no standard congruence tests for quadrilaterals, such as those we all know for triangles. This is an opportunity to explore and discover some results of your own, but there are no set answers that we're after.

ASSESSMENT AND HOMEWORK IDEAS.......

- Assign the "Write and Reflect" concept map or organizational chart for homework.
- Problems 25, 26, 30 and/or the "Checkpoint" problems are all suitable for homework.
- The "Take It Further" problem asks a particularly interesting question about polygons and their diagonals, and would make a good homework assignment.
- Once students have developed a congruence test for rectangles, ask them to make up a proof problem that uses this test.
- Have students write a lab report about their computer investigation.
- Following the constructions, ask students to make up an *impossible* quadrilateral construction, and one that is possible but has more than one answer.
- Plan a test on quadrilaterals and their properties.
- Use the "Write and Reflect" map classifying quadrilaterals for an assessment. Allow students more than one night to work on it, or have them revisit it near the end of the investigation.
- When students complete their computer lab work, grade their written or oral presentations.

WITHOUT TECHNOLOGY . . . .

In this investigation, we have suggested that the "Applications to Analysis and Proof" problems be done on a computer. If the exercise is done without computers, students will move more directly to a theoretical solution, will probably make fewer conjectures, and will have to be encouraged to really look for counterexamples.

Suggest that they make a number of sketches for each problem before making their final conjectures. For example, Problem 21 asks what type of quadrilateral is formed by connecting the midpoints of a rectangle's sides. On the computer, students can easily stretch the figure into many different-shaped rectangles and see that the inner figure remains a rhombus. Students working with paper drawings should sketch several different-shaped rectangles, trying to find any extreme that might change the apparent result.

Congruence in Three Dimensions

Overview

Materials:
- **photocopy of nets**
- **scissors**
- **tape**

The concept of congruence will be well established by the time students reach this investigation, in which students will test this concept by extending it to three-dimensional objects. The main themes are familiar:

- What information about two objects is necessary to prove that they are congruent?
- If we know that two objects are congruent, what does it tell us about the objects?

A set of thought-provoking questions about cubes, cylinders, and pyramids encourages students to exercise habits of mind. Students visualize objects as solids and as *nets* or *developments*, invent congruence tests, and observe what changes occur in a system when one restriction on the system is altered.

Post a list of formulas on the bulletin board for students to check as they work. Volume and surface area formulas are easily forgotten.

This investigation assumes that students have a clear understanding of congruence and how we test for congruence in plane figures. Some questions also require a general knowledge of volume and surface area.

Some problems can be done either theoretically or by actual construction of nets and solids. If you plan to take the hands-on approach, plan to have paper, cardboard, scissors, and tape available.

Teaching the Investigation

Begin this investigation with some visualization exercises. Ask students to picture objects, and then talk about the congruence of the objects. You can use the first seven problems in the Student Module this way, or try one of these:

- Imagine a right triangle. Picture the triangle rotating about one of its sides in such a way that a solid shape is created by the path of the rotation. What shape is formed? Will all the solids made by rotating your triangle be congruent to each other?
- Picture a rectangular box. You want to pack a cylindrical can in the box so that it fits exactly (meeting the box at all sides). If we found all the cans that would possibly fit, would they be all congruent to each other?

Let's assume that she also has tape, but no scissors, and the pieces have to be used whole without folding.

- Jane has six pieces of cardboard that are exactly the size of a regular sheet of notebook paper. Can she make a cardboard box out of them? Follow up: Since the answer is no, now imagine that Jane finds some scissors but she can only

borrow them long enough to cut *two* of the pieces of cardboard. Now she can make a box. But, is there more than one shape box she could make?

The visualizations should spark discussion about what determines congruence for solid objects. Then move to working on the remaining problems in the Student Module. Students can work in small groups. At the conclusion of class, or on the following day, have presentations for the problems that have multiple solutions (for example, Problems 8, 11, and 16).

In the supermarket dairy department, refrigerator rolls are sold in a tube. Check out the way the cylinder unfolds when you open the tube!

A good question to ask: Assuming that one solid figure has many possible nets, do all of its nets have anything in common? What restrictions are there on all nets for a given solid?

ASSESSMENT AND HOMEWORK IDEAS.......

This investigation concludes the module. We assume that teachers will have planned a significant end-of-module assessment to occur at about this time. Suggestions for this can be found in the introductory assessment notes for this module.

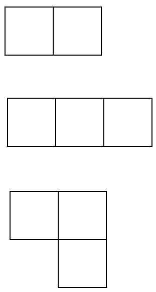
Solutions

Investigation **2.1**

Student Pages 1–3

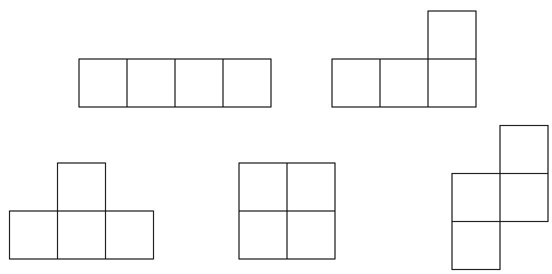
POLYOMINOES

Problems 1–2 *(Student page 1)* Throughout these exercises, you will be realizing that you need to decide upon a precise definition of “the same.”

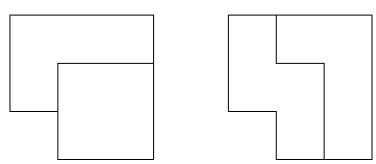


There is only one figure which can be made by combining just two squares. By combining three squares, you can make two different figures: an L shape, or a strip of three squares in a row. One way to find these is to add on one square at a time. Because the first two squares have to obey the polyomino rules, there is just one way to position them. You then need to determine how many different ways you can add on the third square while still following the rules.

Problem 3 *(Student page 2)* Now there are five different shapes which can be made. You can think of this as trying to find how many different ways there are to add two squares to the domino figure, or to add one square to each of the triomino shapes above.

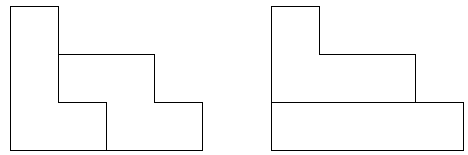


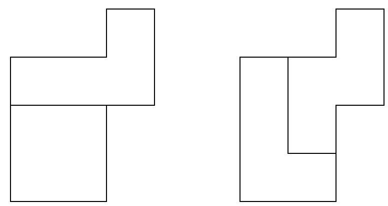
Problems 4–5 *(Student page 2)* Below is an example of a squilch which can be made in two different ways as a combination of an L and another tetromino; these are the only ways to dissect the squilch into such a combination.

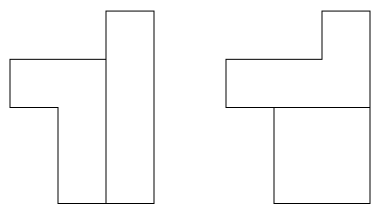


POLYOMINOES *(continued)*

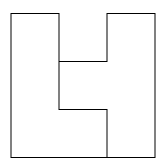
The pictures below show other examples of squilches which can be made in exactly two ways as a combination of an L and another tetromino.







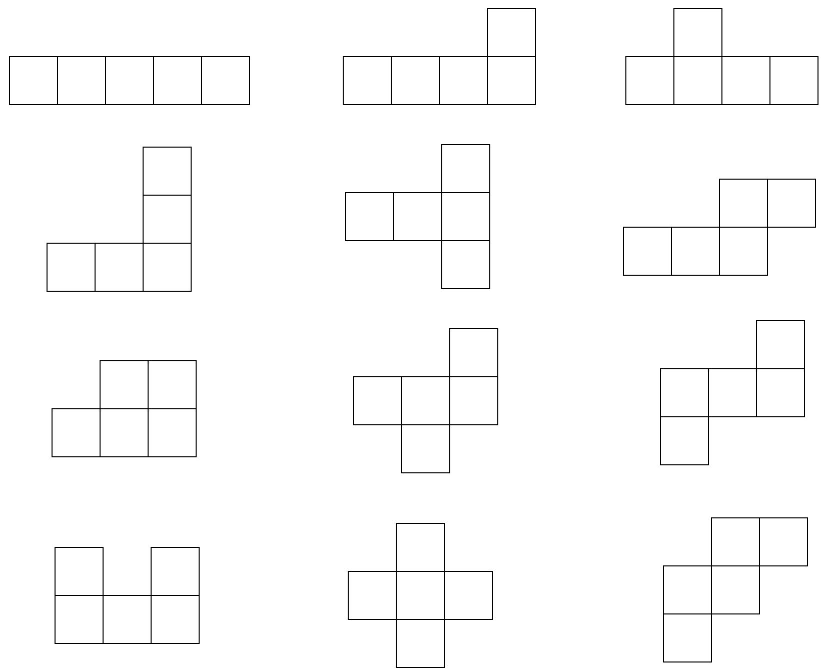
A lot of squilches can be made in only one way as a combination of an L and another tetromino; here's one:



Can you find a squilch which can be made in *three* different ways using an L and another tetromino?

Problem 6 *(Student page 2)* There are 12 different pentominoes. It's tricky to find all 12, but it's best done in a methodical fashion: Work with each tetromino in

turn, finding all the ways to add one square. This will ensure you don't miss any pentominoes, but you have to be careful to avoid duplication. Remember that if a shape can be rotated or reflected so that it matches another, then the two shapes are really the same.



You might find it helpful to have names to refer to each of these 12 pentominoes; one standard bit of notation is to use letters of the alphabet which look similar to each shape. The letters commonly used are T, U, V, W, X, Y, Z, F, L, I, P, and N. In the diagram above, the ordering of these figures is as follows:

I	*L*	*Y*
V	*T*	*N*
P	*F*	*Z*
U	*X*	*W*

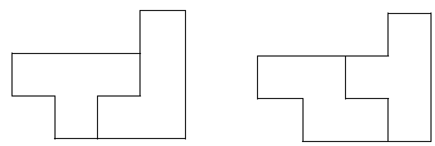
You might have to reflect or rotate some of the shapes in order to match them to their letter name.

Problem 7 *(Student page 2)* All the terms end in "omino", but the first part of the word is a prefix indicating how many squares are used. A polyomino made from 10 squares would be called a *decomino*.

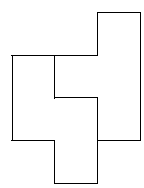
POLYOMINOES (continued)

Problem 8 *(Student page 2)* If you cut out two pentominoes, they will be the same if you can rotate one so that it fits perfectly on top of the other.

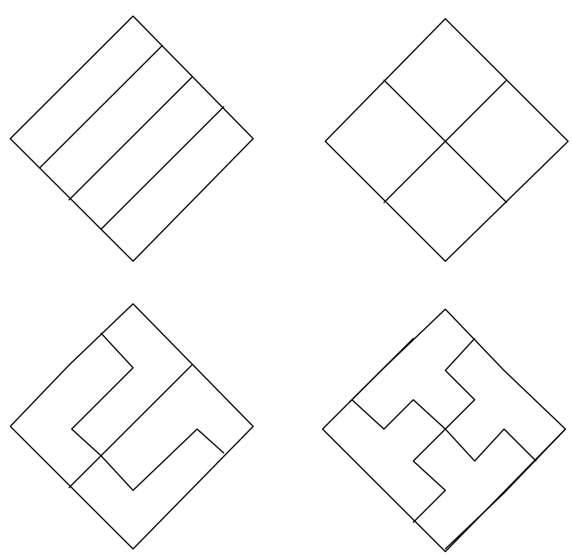
Problems 9–10 *(Student page 2)* Here's an example of a T-squilch which can be made in two different ways:



and one which can be made in only one way:

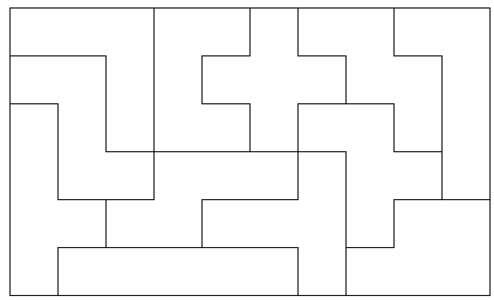


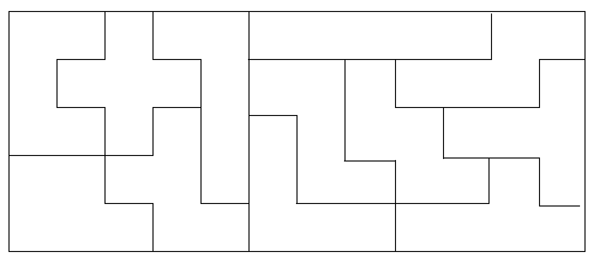
Problem 11 *(Student page 2)* The picture below shows the four tetrominoes which do cover an 8 × 8 checkerboard; the one shape which doesn't work is the "skew" tetromino.



Problem 12 *(Student page 3)* Only the straight and the L tetrominoes can do this. It's difficult to prove that the other two won't work, so it's enough to just find the correct two.

Problem 13 *(Student page 3)* Here are two possible solutions; one for the 6×10 rectangle, and one for the 5×12.



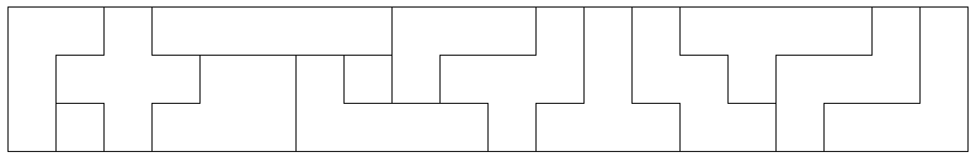


The 6×10 has 2339 possible solutions, and the 5×12 has only 1010 possible solutions.

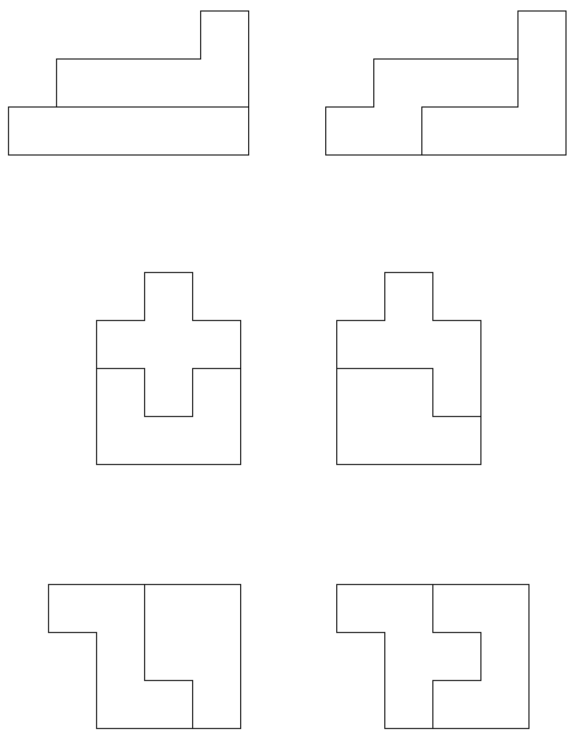
Problem 14 *(Student page 3)* It's not possible to arrange the 12 pentominoes into a 2×30 rectangle. This is because many of the pentominoes are three squares deep in both length and height, so there is no way to position them so that they would fit inside the rectangle.

POLYOMINOES (continued)

Here's an example of one of the two arrangements of the pentominoes into a 3 × 20 rectangle, as referred to after Problem 14 in the Student Module.

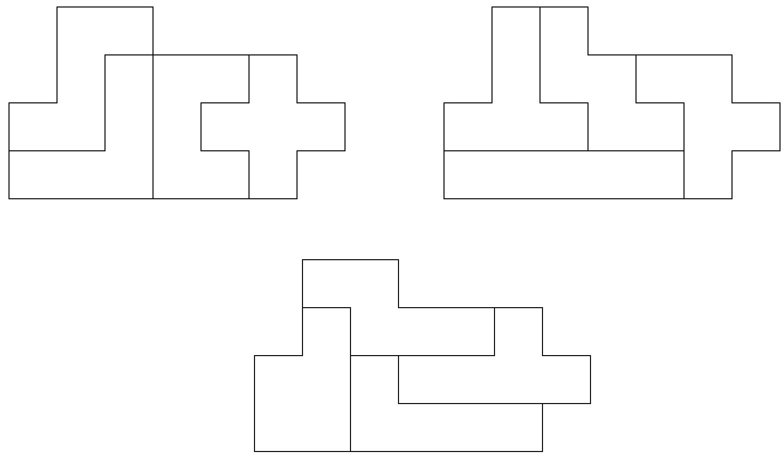


Problem 15 *(Student page 3)* There are many ways to arrange four different pentominoes into two identical ten-square shapes. A few of these ways are shown below.



POLYOMINOES (continued)

Problem 16 *(Student page 3)* Below are three identical shapes, each consisting of four different pentominoes.



Comparing Pictures

For Discussion *(Student page 4)* All of the figures are squares, so in some sense they are all the same. Yet, the lower left square has smaller area than the others, while the upper right square is oriented differently. Using the idea that two figures are the same if one can be rotated so that it fits perfectly on top of the other, we can conclude that all of the squares except for the lower left are the same; they are all congruent.

Problem 2 *(Student page 6)* The following pairs of triangles might be congruent: $\triangle AFD$ and $\triangle AEB$, $\triangle FBD$ and $\triangle EDB$, or $\triangle EGD$ and $\triangle FGB$.

Problem 3 *(Student page 6)* The shapes in each pair seem congruent. One way to see this is to cut out both shapes in a pair and see if you can fit one on top of the other. You can also compare measurements. For example, both of the circles have a radius of 0.5 inches, and so are congruent, and both pentagons are regular, with all sides of length 0.6 inches, and so they are also congruent. You might also measure the distances between corresponding vertices in both the stars and snowflakes, or the lengths of corresponding segments of the arrows.

Notice that although the cats are congruent, one picture will have to be flipped over to fit on top of the other.

Problem 4 *(Student page 8)*

- The coffee cup on the upper right is different because the steam is rising away from the handle, while the other cups have steam rising toward their handles.
- The door knocker in the upper left corner has larger "spines" than the other three.
- The soccer ball in the lower right corner has a smaller diameter than the others.

Problem 5 *(Student page 9)* Some strategies include measuring, tracing, or cutting the figures, as well as inspecting them visually.

Problem 8 *(Student page 9)* All equilateral triangles *with the same sidelength* are congruent.

Problem 9 *(Student page 10)* One way to see whether the two figures are congruent is to cut them out, place one on top of the other, and see if they coincide. If your definition is that two shapes are congruent if all corresponding features agree, you can measure the sides and angles of each polygon.

COMPARING PICTURES *(continued)*

Problem 10 *(Student page 10)* The goal of this problem is to start thinking about the characteristics of a good definition, as we will later be interested in formulating good mathematical definitions. A good definition should convey enough information, but not so much so as to be overwhelming. It should be precise, with not a lot of extraneous details. Another quality of a good definition is that after reading it you are able to decide whether or not a given object satisfies it. For example, the definition of a square should contain enough information so that you can easily tell that not all rectangles are squares.

One way to approach this problem is to first try to write your own definitions for the given words, and then compare what you wrote with what is in the dictionary. Which definition do you prefer?

Problem 11 *(Student page 11)* See whether this process will lead you to simpler and simpler words, or whether you will end up in a chain of circular reasoning. It might be fun to try this activity with a technical word whose definition you don't know well. If the dictionary gives many definitions for a word, just stick to the primary one.

An example of circular definitions is found by looking up the word *belong*, which leads to the word *proper*, which leads to *appropriate*, which leads to *suitable*, but it turns out that *appropriate* is used in the definition of *suitable*!

THE CONGRUENCE RELATIONSHIP

Note that whether or not two angles are congruent has nothing to do with the length of their sides.

Problem 1 *(Student page 12)* Two line segments are congruent if they have the same length or if they exactly cover each other. Similarly for angles; you are likely to define two angles to be congruent if they have equal measure or if one can exactly cover the other. Two triangles are congruent if they have three pairs of congruent corresponding sides and three pairs of corresponding congruent angles. Two solids can also be considered congruent if they have the same size and shape; one way to measure size is by volume.

Problems 2–3 *(Student page 12)* You will likely agree that the cut-and-move test will work for line segments; what does it mean to "cut out" a line segment? In extending the superimposability test for congruence to three-dimensional objects, you can imagine that two 3D objects are congruent if they have the same size and shape and take up the same space. One conceivable test for 3D congruence is to wrap a sphere or box and see if the possibly congruent shape fits into that same wrapper. Maybe you could see whether both objects hold the same amount of water. Or you could fill the object with sand, submerge it in water, and then check the water's displacement.

Problem 4 *(Student page 12)* If two polygons are congruent, they must have the same area. This is what we mean when we say that we can place one exactly on top of the other. However, just because two polygons have the same area, they are not necessarily congruent. A $4'' \times 4''$ square has the same area as an $8'' \times 2''$ rectangle, but they are not congruent.

Problem 5 *(Student page 13)*

a. The length of segment $\overline{JK}$ equals the length of segment $\overline{RS}$. Two numbers are being compared here.

b. Segments $\overline{JK}$ and $\overline{RS}$ are congruent.

c. This is the same as $JK = RS$. Again, two numbers representing the lengths of segments are being compared.

d. This also says that the length of segment $\overline{JK}$ equals the length of segment $\overline{RS}$. The distance between two points means precisely the length of the straight line segment connecting them. (The shortest path between two points is along a straight line.)

Problem 6 *(Student page 14)*

- **a.** This statement doesn't make sense. JK and RS refer to lengths, while the congruence symbol is only defined for shapes.
- **b.** This is correct; the sentence says that two line segments are congruent.
- **c.** This is also correct; it is OK to say that two numbers are equal. This statement says that the length of $\overline{JK}$ equals the length of $\overline{RS}$.
- **d.** This is false. Two line segments can't be equal; they can only be congruent.
- **e.** This is fine. The statement says that the length of $\overline{JK}$ is 1, or that the distance from J to K is 1.
- **f.** This is incorrect. A segment can't equal a number.
- **g.** This is incorrect. By 1″, you mean a measurement, and a segment can't be congruent to a measurement.

To form the converse of a statement, you switch the "if" and "then" parts.

Statement: If you live in Ohio, then you live in the United States.

Converse: If you live in the United States, then you live in Ohio.

Is the converse of a true statement always true?

Problems 7–8 *(Student page 14)* All segments have the same "shape," so if they have the same length, one can be superimposed on the other. Therefore, segments with the same length are congruent. For the converse, if two segments are congruent, they definitely have the same length.

Problem 9 *(Student page 15)* The sentence "$\angle NPQ = 56.6^\circ$" is confusing. The symbol $\angle NPQ$ refers to the object which is the angle, not the number which is the measure of the angle. What is meant here is that *the measure* of the object $\angle NPQ$ is 56.6°. This is written as $m\angle NPQ = 56.6^\circ$.

Problems 10–11 *(Student page 15)* Yes, if two angles have the same measure, they are congruent. If you try the "superimposability test," the sides of the angles should lie exactly on top of each other, but if the sides are segments (rather than rays), they may not have the same length. Moreover, if two angles are congruent, they match when placed on top of each other, so they have the same measure.

Problem 12 *(Student page 15)*

- **a.** Two segments are congruent if they have the same length.
- **b.** Two angles are congruent if they have the same measure.
- **c.** Two triangles are congruent if all corresponding sides and angles have the same measurements.
- **d.** Two cubes will be congruent if they both have the same length edges, for

then they will both have the same shape (a cube), and take up the same amount of space, which is a way to think of congruency for 3D objects.

Problem 13 *(Student page 15)*

a. True; since $AD = BD$, it follows that $\frac{1}{2}AD = \frac{1}{2}BD$, so $FD = DE$ since F and E are midpoints.

b. Nonsensical; two segments can't be equal, they can only be congruent.

c. Nonsensical; a segment can't equal a measurement.

d. Nonsensical; you can say that $\angle ACD$ is a right angle or that $m\angle ACD = 90°$.

e. Nonsensical; a triangle can't equal an angle!

f. True; an angle which measures $90°$ is a right angle.

g. True; by the same reasoning as in the first statement, we know that $\overline{FA}$ and $\overline{BE}$ have the same length, and hence are congruent.

h. False; two segments can't be congruent unless they have the same length. We know that $FA = \frac{1}{2}DA$, but $DA = BD$, so $\overline{FA}$ is half the length of $\overline{BD}$. Therefore, $\overline{FA}$ and $\overline{BD}$ are not congruent.

i. Nonsensical; two angles cannot be equal, they can only have the same measure, which would make them congruent.

j. True; the altitude of an isosceles triangle bisects the vertex angle. You can rely on measurements if you don't know this.

k. Nonsensical; two measurements cannot be congruent.

l. True; since you haven't worked on proving triangles congruent yet, you can just measure for now.

m. Nonsensical; a triangle can't be congruent to an angle.

n. Nonsensical; once again, a triangle can't be congruent to an angle!

Problem 14 *(Student page 16)* True; this is the transitive property of congruence. What this means is that all three triangles have the same size and shape, and all three can fit perfectly on top of each other. Think of it this way: if A fits perfectly on B, and B fits perfectly on C, then A must fit perfectly on C.

Problem 15 *(Student page 16)* This problem introduces the mathematical notation used to identify line segments of equal length and angles of equal measure. Segments with the same marks are congruent, as are angles with the same marks.

You might find all the uses of transitivity.

Problem 16 *(Student page 17)* The following line segments and angles are congruent:

- $\overline{AB} \cong \overline{LM}$, $\overline{AB} \cong \overline{OP}$, $\overline{LM} \cong \overline{OP}$
- $\overline{FE} \cong \overline{GH}$, $\overline{GH} \cong \overline{IH}$, $\overline{FE} \cong \overline{IH}$
- $\overline{BC} \cong \overline{ON}$
- $\angle DEF \cong \angle NOP$
- $\angle ABC \cong \angle GHI$

Problem 17 *(Student page 17)* The correct statement is d, $\triangle DFA \cong \triangle ECG$. When writing congruences such as this, it is customary to list corresponding vertices in order. In the two triangles, angles D and E are congruent, as are F and C, as well as A and G. So the vertices DFA in order correspond to the vertices ECG. Moreover, the order in which the vertices are listed also gives information about the sides of the triangle. For instance, if $\triangle DFA \cong \triangle ECG$, then you immediately know that $\overline{DF} \cong \overline{EC}$, $\overline{FA} \cong \overline{CG}$, and $\overline{AD} \cong \overline{GE}$, just from the order of the vertices.

There are six correct statements, each obtained by permuting (reordering) the vertices.

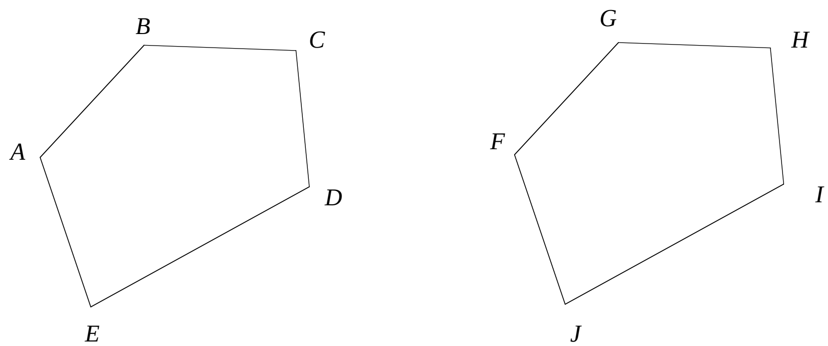
Problem 18 *(Student page 17)* The congruences are:

- $\triangle DFA \cong \triangle ECG$
- $\triangle FAD \cong \triangle CGE$
- $\triangle AFD \cong \triangle GCE$
- $\triangle DAF \cong \triangle EGC$
- $\triangle FDA \cong \triangle CEG$
- $\triangle ADF \cong \triangle GEC$.

Notice that there is an alternate answer. The statement $\triangle DFA \cong \triangle ECG$ also yields the following six statements regarding the sides and angles of the two triangles:

- $\overline{FD} \cong \overline{CE}$
- $\overline{FA} \cong \overline{CG}$
- $\overline{FD} \cong \overline{EC}$
- $\angle D \cong \angle E$
- $\angle F \cong \angle C$
- $\angle A \cong \angle G$.

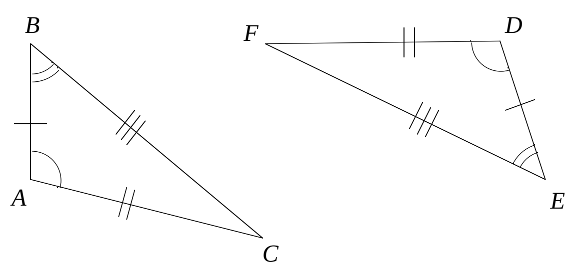
Problem 19 *(Student page 18)* In the picture below, pentagons *ABCDE* and *FGHIJ* are congruent.



The congruent sides are: $\overline{AB} \cong \overline{FG}$, $\overline{BC} \cong \overline{GH}$, $\overline{CD} \cong \overline{HI}$, $\overline{DE} \cong \overline{IJ}$, and $\overline{EA} \cong \overline{JF}$.

The congruent angles are: $\angle A \cong \angle F$, $\angle B \cong \angle G$, $\angle C \cong \angle H$, $\angle D \cong \angle I$, and $\angle E \cong \angle J$.

Problem 20 *(Student page 18)* Here's an example of a pair of congruent triangles and the corresponding congruence statements.



Do you know that $\angle C \cong \angle F$ even though they have no markings?

The phrase "corresponding parts" will be used frequently.

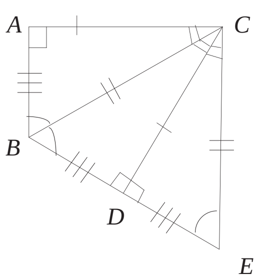
Triangles $\triangle ABC$ and $\triangle DEF$ are congruent, with the following pairs of corresponding parts congruent: $\angle A \cong \angle D$, $\angle B \cong \angle E$, $\angle C \cong \angle F$, $\overline{AB} \cong \overline{DE}$, $\overline{AC} \cong \overline{DF}$, and $\overline{BC} \cong \overline{EF}$.

Problem 21 *(Student page 18)* Below is a sample answer:

> If two figures are congruent, then one is really just a copy of the other, although it may be flipped, rotated, or translated. That means that every angle, segment, or point on the first figure has a "twin" on the second

figure. These associated pairs are called *corresponding parts*. If two figures are congruent, you can match up these corresponding pairs and they wlll be congruent to each other.

Be careful about angles, though. Remember that whether two angles are congruent has nothing to do with the size of the sides that form the angles. This may seem like a contradiction, but, in fact, angles are usually defined as having rays for sides, so there is no length to worry about.

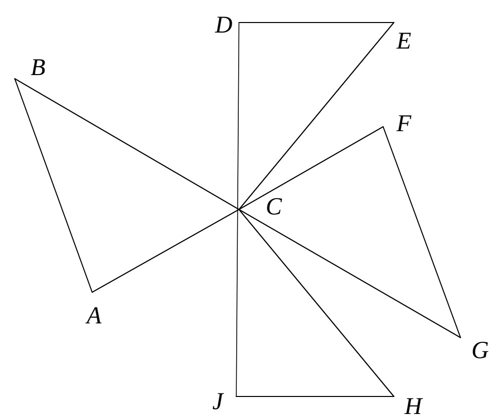


The sketch, with markings

Problem 22 *(Student page 19)* The three congruent triangles are $\triangle ABC$, $\triangle DBC$, and $\triangle DEC$. In the kite *ABDC*, $\triangle DBC \cong \triangle ABC$, and in triangle $\triangle BCE$, $\triangle BCD \cong \triangle ECD$. Notice that congruent triangles are always listed with corresponding vertices in order.

Problem 23 *(Student page 19)* Because $\angle E \cong \angle H$, and $m\angle H = 50°$, it follows that $m\angle E = 50°$. Also, since $\angle A \cong \angle F$, and $m\angle F = 80°$, you know that $m\angle A = 80°$. Since $m\angle B = 40°$, then $\angle JCH$ is also $40°$, since the two angles have the same markings. Notice that $\angle JCH \cong \angle DCE$, as they have the same marking, so $m\angle D = 180° - 50° - 40° = 90°$.

Notice this picture looks quite different from the one in the Student Module.



The corrected picture

Solutions

Investigation **2.4**

Student Pages 20–21

STRONG LANGUAGE

You might compose some examples of nonmathematical strong statements.

For Discussion *(Student page 20)* Such a strong statement could *not* be made on the basis of measurement or superimposition. A strong statement is one which holds for *all* objects satisfying the hypotheses of the statement. There is no possible way to measure all objects of any given type, so the statement has to be justified mathematically, not by measuring one particular figure. For example, the statement, "the sum of the measures of all angles in a quadrilateral is 360°" is a strong statement which applies to all quadrilaterals. You can't justify the statement by measuring the angles in just one square, for instance.

In the second part of the discussion, Statement 1 is not a strong statement; it is about a particular picture, and thus can be verified by measuring. Statement 2, on the other hand, is a stronger statement, as it makes an assertion about *any* angle inscribed in *any* semicircle.

Problem 1 *(Student page 21)* The first statement could be verified by precise measurement, as it is about four particular triangles which are drawn on the page. The second statement is stronger, as it applies to *any* right triangle. Mathematicians might say that the second statement is a generalization of the first.

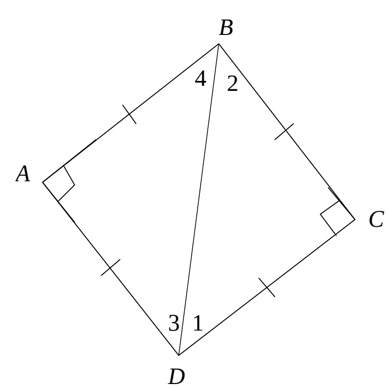
Triangle Congruence

For Discussion *(Student page 22)* This test for congruence certainly agrees with the idea that two triangles are congruent if they differ only in position or orientation. For if they differ only by position, then one can fit exactly on top of the other, meaning precisely that all pairs of corresponding parts match each other, so they must be congruent. On the other hand, if all pairs of corresponding parts are congruent, then you can match them up in such a way so that one triangle fits on top of the other. The two ideas are equivalent.

Problem 1 *(Student page 22)* Given any square, you can cut along the diagonal, producing two right triangles. One of the triangles can be reflected about the diagonal and will fit exactly on top of the other.

Here is a second way to show the two triangles are congruent: Suppose $ABCD$ is a square, with diagonal $\overline{DB}$. Label angles 1, 2, 3, and 4 as in the picture below. By the definition of a square, you know that there are four congruent sides. Specifically $\overline{AB} \cong \overline{CD}$ and $\overline{AD} \cong \overline{CB}$. Notice that triangles $\triangle ABD$ and $\triangle CBD$ share side $\overline{BD}$. Thus, all their corresponding sides are congruent.

Now look at the angles. You know that $\angle A \cong \angle C$, since they are both right angles. Because $\overline{AB}$ is parallel to $\overline{CD}$, $\overline{DB}$ is a transversal, with angles 1 and 4 becoming alternate interior angles; hence $\angle 1 \cong \angle 4$. Similarly, $\angle 2 \cong \angle 3$, since $\overline{AD}$ is parallel to $\overline{CB}$. Therefore, all corresponding sides and angles are congruent, so $\triangle ABD \cong \triangle CDB$.



There is a second way to see that $\angle 1 \cong \angle 4$ and $\angle 2 \cong \angle 3$. Because the sum of the measures of the angles in a triangle is 180°, you know that $m\angle 1 + m\angle 2 = 90^\circ$ and $m\angle 3 + m\angle 4 = 90^\circ$.

Because $\angle ADC$ and $\angle ABC$ are right angles, you know that $m\angle 1 + m\angle 3 = 90^\circ$ and $m\angle 4 + m\angle 2 = 90^\circ$.

Solving for $m\angle 1$ in the first and third equations above yields

$$m\angle 1 = 90^\circ - m\angle 2 = 90^\circ - m\angle 3,$$

implying that $m\angle 2 = m\angle 3$.

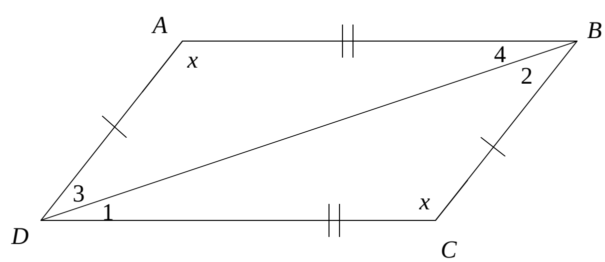
Solving for $m\angle 2$ in the first and fourth equations above yields

$$m\angle 2 = 90^\circ - m\angle 1 = 90^\circ - m\angle 4,$$

implying that $m\angle 1 = m\angle 4$.

Problem 2 *(Student page 22)* The diagonal of a rectangle does divide the rectangle into two congruent triangles. The same argument as above works; in the case of the square, we never used the fact that all four sides were of equal length, only that opposite sides were congruent.

The statement is also true for parallelograms, and can be proved similarly. Suppose you have a parallelogram as in the picture below.



It's a good habit to try to use what you have already proved, if possible.

Opposite sides of a parallelogram have the same length, so using the above argument once again we know that all pairs of corresponding sides of the two triangles are congruent: $\overline{DA} \cong \overline{BC}$, $\overline{AB} \cong \overline{CD}$, and, of course, $\overline{BD} \cong \overline{BD}$.

In a parallelogram, opposite angles are congruent, while consecutive angles are supplementary. Therefore $\angle A \cong \angle C$; suppose their value is x°. Since consecutive angles are supplementary, you know that

$$1. \quad m\angle 1 + m\angle 3 + x = 180^\circ$$

and

$$2. \quad m\angle 2 + m\angle 4 + x = 180^\circ.$$

Since *ABD* and *CDB* are triangles, the sum of the measures of their angles is 180°, so

$$3. \quad m\angle 1 + m\angle 2 + x = 180^\circ$$

and

$$4. \quad m\angle 3 + m\angle 4 + x = 180^\circ.$$

Using equations 1 and 3,

$$m\angle 1 + m\angle 3 + x = m\angle 1 + m\angle 2 + x = 180^\circ,$$

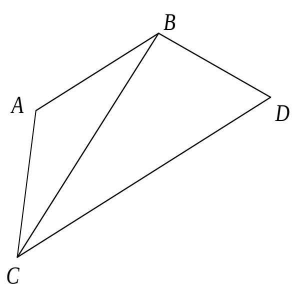
so $m\angle 3 = m\angle 2$.

Using equations 1 and 4,

$$m\angle 1 + m\angle 3 + x = m\angle 3 + m\angle 4 + x = 180^\circ,$$

so $m\angle 1 = m\angle 4$.

You now know that all corresponding pairs of angles are congruent, so you can conclude that $\triangle ABD \cong \triangle CDB$.



In a trapezoid, it is not true that a diagonal cuts the figure into two congruent triangles. There is a relation between the angles in a trapezoid; namely, that in the side picture,

$$m\angle A + m\angle ACD = m\angle ABD + m\angle D = 180^\circ,$$

but this relation is not sufficient to guarantee that a diagonal produces two congruent triangles.

For Discussion *(Student page 23)* You aren't by any means expected to find all the congruence tests here. The point of the problem is to just get you started thinking about whether or not there are any shortcuts to testing all six corresponding pairs. One easy suggestion is that the third pair of angles never has to be checked if you already know that the first two pairs are congruent, because the measure of the third angle in each triangle will be 180° minus the sum of the other two, which are known to be congruent.

It is true that information which is sufficient to specify one triangle exactly is also enough to ensure that two triangles are congruent. In proving that two triangles are congruent, you are trying to show that they are essentially the exact same triangle.

Problem 3 *(Student page 24)* You will need either three or four pieces of information to determine the triangle. Answers will vary, depending on both the information given and the order that it is drawn from the envelope. The two combinations AAA and SSA will never work to describe a unique triangle, but both will work if you add a fourth piece of information.

Problem 4 *(Student page 24)* You will never need to draw more than four notes to construct a unique triangle. An example of such a "worst case" would be if you drew three angle measurements, followed by a sidelength. Often three notes will allow you to construct a triangle, but the triangle will not be unique (such as AAA).

Problem 5 *(Student page 24)* Having four pieces of information guarantees that any triangle you construct will be unique, no matter which combination of four you choose. Many combinations of three measurements will work, but not all of them. If the three measurements are all angle measurements, for example, you can get an infinite number of triangles all with those angles. Two pieces of information will never be enough to guarantee a unique triangle.

For Discussion *(Student page 24)* The "magic number" is 3, but whether you get a unique triangle or a variety of triangles depends on which combination of three pieces of information you get.

Problem 6 *(Student page 25)* There are 20 different combinations of 3 pieces of information, if the order in which you choose them doesn't matter. The 20 possibilities in this case are:

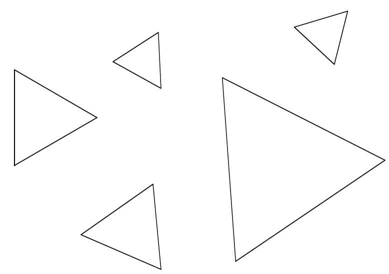
If the order mattered in choosing 3 combinations (i.e., if $m\angle A$, $m\angle B$, $m\angle C$ were different from $m\angle B$, $m\angle C$, $m\angle A$), then there would be 720 combinations!

$m\angle A, m\angle B, m\angle C$	$m\angle A, m\angle B, AB$
$m\angle A, m\angle B, BC$	$m\angle A, m\angle B, AC$
$m\angle A, m\angle C, AB$	$m\angle A, m\angle C, BC$
$m\angle A, m\angle C, AC$	$m\angle A, AB, BC$
$m\angle A, AB, AC$	$m\angle A, BC, AC$
$m\angle B, m\angle C, AB$	$m\angle B, m\angle C, BC$
$m\angle B, m\angle C, AC$	$m\angle B, AB, BC$
$m\angle B, AB, AC$	$m\angle B, BC, AC$
$m\angle C, AB, BC$	$m\angle C, AB, AC$
$m\angle C, BC, AC$	AB, BC, AC

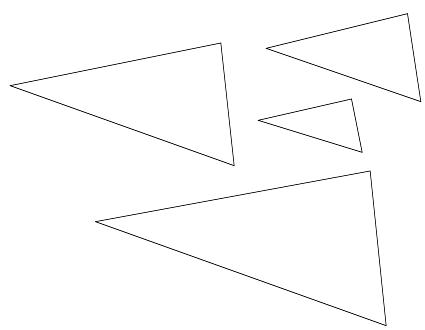
Problem 7 *(Student page 26)* The "good" codes are AAS, ASA, SAS, and SSS, while the "bad" ones are AAA and SSA. The pictures below illustrate why AAA is

TRIANGLE CONGRUENCE *(continued)*

a "bad" code; all of the triangles below have each of their angles measuring 60° (as will any equilateral triangle), but the triangles are not congruent:

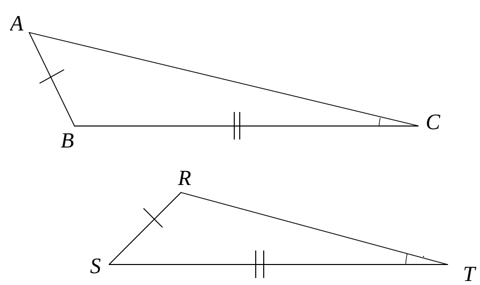


Each of the triangles in this picture consists of angles measuring 30°, 65°, and 85°, yet the triangles are not congruent:



Another "bad" code is SSA; if you draw two sidelengths and an angle measurement from the envelope, you are not guaranteed a unique triangle. For example, suppose you draw two sides, of length 1.5 cm and 5 cm, along with an angle measurement of 15°. Both of the triangles below could be made with such a combination, but the triangles are distinctly different, so they are not congruent.

This is why there is not an SSA test for congruent triangles.



Problem 8 *(Student page 27)* There is not enough information, as only two sides are given.

Problem 9 *(Student page 28)* The SSS postulate can be applied here. The congruence is given by $\triangle ABC \cong \triangle DEF$. (Remember that by listing corresponding vertices in the proper order, you immediately know exactly how the triangles match up.)

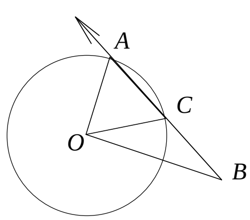
Problem 10 *(Student page 28)* Use the SAS postulate to conclude that $\triangle ABC \cong \triangle DBC$. (Notice that $\overline{BC}$ is a common side.)

Problem 11 *(Student page 28)* While the triangles may in fact be congruent, you can't determine this from the information given. In $\triangle ABC$, you are given two sides and the angle between them, but in $\triangle BCD$ you are given two sides and a non-included angle. To prove that two triangles are congruent, you need information about corresponding parts of the two triangles, which you don't have in this example.

Problem 12 *(Student page 29)* Yes, these triangles are congruent. Use the two pairs of marked angles and the common side $\overline{AC}$ to conclude that $\triangle ABC \cong \triangle CDA$ by the ASA postulate.

Problem 13 *(Student page 29)* Yes, $\triangle ABC \cong \triangle YXC$ by the SAS postulate.

Problem 14 *(Student page 30)* It is ambiguous whether the 14-inch and the 8-inch sides are the two sides of the triangle adjacent to the 30° angle at the tip of the pennant, or if one of those two sides is opposite the 30° angle. This makes a difference, because knowing two sides and the angle between them is sufficient to determine a unique triangle, but knowing two sides and the non-included angle is not.



Problem 15 *(Student page 30)* Look at triangles $\triangle ABO$ and $\triangle OBC$ in the picture. Sides $\overline{AO}$ and $\overline{CO}$ are congruent, as they are both radii of the circle. Both triangles also share side $\overline{OB}$ and angle $\angle OBC$. The two triangles are definitely not congruent, however, as one is contained inside the other. Thus, knowing SSA information is not enough to determine a triangle uniquely.

Problem 16 *(Student page 30)* One way to disprove the conjecture is to take two equilateral triangles, each with a different sidelength. They won't be congruent, as they will be different sizes, but they will have all corresponding angles of equal measure (as all three angles in both triangles will be 60°).

Problem 17 *(Student page 30)* The idea that many examples won't prove a conjecture but one counterexample will disprove it is tricky but very important. It may

help to think of nonmathematical situations at first. Here is one example: Suppose you have the conjecture that it always rains on Tuesdays. Even if you've observed it rain for 10 Tuesdays in a row, or even for 100 Tuesdays in a row, you cannot conclude that it *always* rains on Tuesday. However, if on just one particular Tuesday it does not rain, then you know right away that the conjecture fails. Once you see some examples like this, it is easier to see that the same reasoning holds for mathematical statements.

Problem 18 *(Student page 30)* Triangles $\triangle ABD$ and $\triangle CBD$ are congruent. The fact that $\overline{BD}$ is the perpendicular bisector of $\overline{AC}$ yields two pieces of information:

$$AD = DC, \text{ and } m\angle BDA = m\angle BDC = 90^\circ.$$

This, combined with the fact that both triangles share side $\overline{BD}$ lets you apply the SAS postulate.

Problem 19 *(Student page 31)*

a. Adding that $AB = AC$ gives no new information. Just knowing that $\overline{AD}$ is the perpendicular bisector of $\overline{BC}$ gives us that $\triangle ABD \cong \triangle ACD$, and from this we automatically conclude that $AB = AC$ by CPCTC. Hence, we are given no new information, and cannot prove any new triangles congruent.

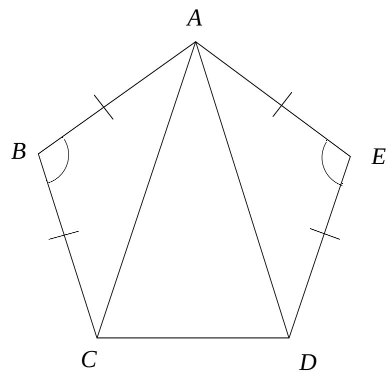
b. You can prove that $\triangle ADE \cong \triangle ADF$. Because $\overline{AD}$ is the perpendicular bisector of $\overline{EF}$, we know that $ED = FD$. This, combined with what you already knew, gives the congruence using SAS.

You can also prove that $\triangle AEB \cong \triangle AFC$ and $\triangle AEC \cong \triangle AFB$. The two known congruences, $\triangle ADB \cong \triangle ADC$ and $\triangle ADE \cong \triangle ADF$, yield $DB = DC$ and $DE = DF$ by CPCTC. These statements imply that $BF = EC$ and $BE = FC$. CPCTC also gives that $AB = AC$ and $AE = AF$. Now apply SSS to obtain the two congruences.

c. $\triangle AED \cong \triangle AFD$. This time the congruence is given by ASA, combining the fact that $m\angle EAD = m\angle FAD$ with what was already known.

Problem 20 *(Student page 31)* In regular polygons other than quadrilaterals, drawing all the diagonals from one vertex does not produce triangles which are all congruent. For example, in the regular pentagon shown here, all the diagonals from vertex A are drawn. $\triangle ABC$ and $\triangle AED$ are congruent by the SAS postulate (since

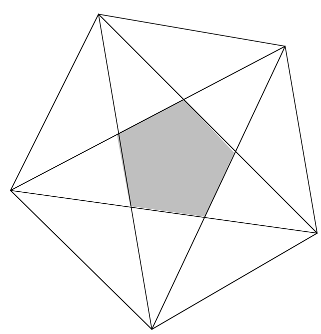
are not congruent to $\triangle ACD$, so the diagonals did not divide the pentagon into triangles that are all congruent to each other.



Similar things will happen for polygons with more than five sides: you will get pairs of congruent triangles (with one in the middle if you have an odd number of sides).

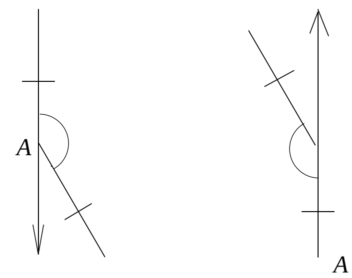
Problem 21 *(Student page 31)* Here is another counterexample involving a regular pentagon. Notice that, if all the diagonals are drawn, they form a smaller pentagon in the center of the original figure (see the picture below). This polygon does not have a congruent match anywhere in the larger pentagon.

Conjecture: If you start with a regular polygon with an even number of sides, then each polygon formed *will* have a match.



Problem 22 *(Student page 31)* The first set of Logo commands has you move forward 20 units, then back 20 units to where you started, followed by a turn of 150° to the right, and then you head out 20 units in this new direction. In the second set of commands you move forward 20 units, turn 30° to the left, and move forward 20 units in that direction. This sounds as if it would produce two different figures, but if you draw them, you will see that they are, in fact, congruent. In each of the following

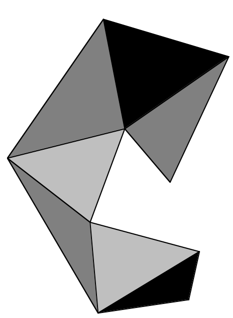
pictures, the starting location is labeled A; notice that if you reflect the second figure, you can place it on top of the first.



Problem 23 *(Student page 31)* If the planes intersect the cube so that they are both parallel to one of the faces of the cube, then the cross sections will be congruent to each other, and to the side of the cube. Moreover, the cross sections will also be congruent whenever the two parallel planes are equidistant from the center of the cube.

Problem 24 *(Student page 32)* If you add 2 to each coordinate of D, E, and F, you will get a new triangle which is congruent to $\triangle DEF$. One way to see this is by the SSS postulate: the triangle has been translated, but each side is still the same length. You can verify this using the distance formula for points in the plane.

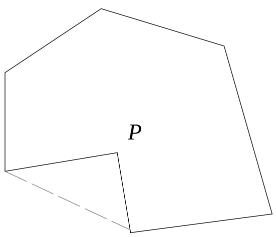
If you multiply each coordinate by 2, however, your new triangle will not be congruent to the original; it will be much bigger.



Problem 25 *(Student page 32)* It is true that all polygons can be cut up into triangles. You should start to believe this after playing with many examples. One way to think of it is to start with three adjacent vertices of the polygon. These vertices can be connected to form a triangle, and you can imagine cutting off this triangle from the polygon, forming a new polygon with fewer sides. This process can be continued, yielding new polygons with fewer and fewer sides. You will eventually come to a stop, when the new polygon you form is a triangle itself. This is an example of an induction argument.

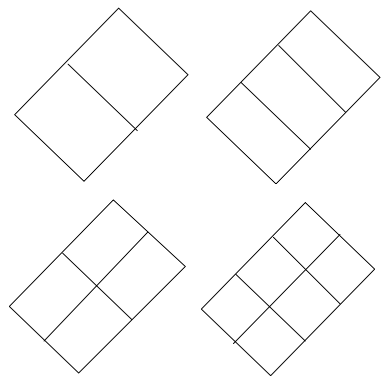
This also shows that, if you start with a polygon with n sides, you will get $n-2$ triangles. The first and last triangles formed as above will each contain two of the polygon's sides. All of the other triangles will contain one side each.

In a *concave* polygon, at least one diagonal falls outside the figure. In the figure below, *P* is called a *point of concavity*.

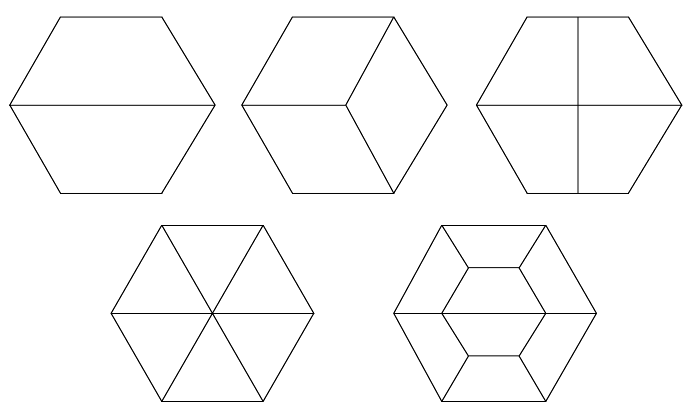


One warning about this algorithm is necessary, however. If you start by choosing three vertices which form a concavity, then the triangle they form will not be inside the original polygon, but rather on the outside. In this case, you need to start with a different set of three vertices. This isn't a problem, though, as it's not possible for every vertex of the polygon to be a point of concavity.

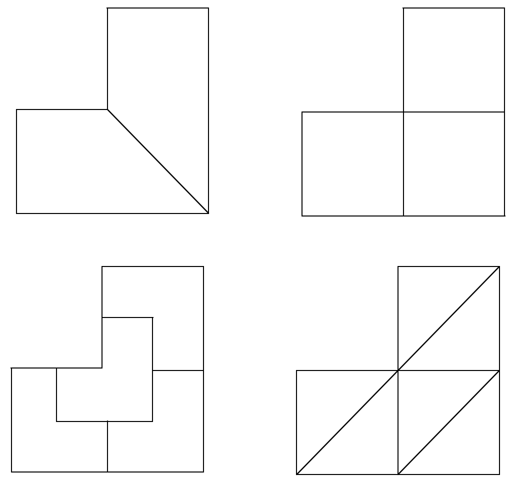
Problem 26 *(Student page 33)* The easiest way to do this problem is to simply divide the rectangle into smaller congruent rectangles.



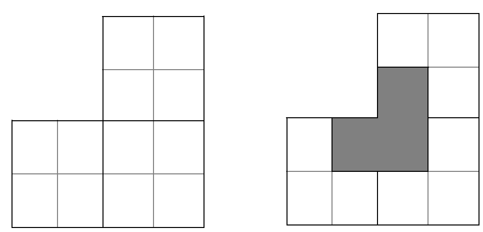
Problem 27 *(Student page 33)* See the figures below. It's easy to divide the hexagon into two or four congruent pieces. To divide it into three congruent pieces, connect the center to alternating vertices, while to divide it into six pieces, connect the center to each vertex. Getting eight congruent pieces is tricky. First, connect the center to each vertex, and then construct an internal hexagon by connecting the midpoints of these segments. This smaller hexagon can be divided into two trapezoids, and the remainder of the original hexagon is divided into six more congruent trapezoids.



Problem 28 *(Student page 34)* The hardest part here is dividing the figure into four congruent pieces. What you need to do is to divide the shape into four smaller L-shaped pieces.

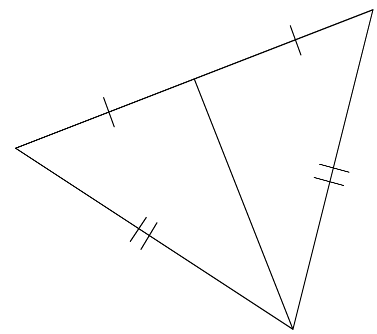


You can think of going from the case of three pieces to the case of four pieces by dividing each of the three squares into four smaller squares, then taking three of these squares for each of the four congruent pieces.



Problem 29 *(Student page 35)* The given triangle is isosceles, and the median to the noncongruent side creates two congruent triangles. Since a median is a segment

that bisects a side and the two triangles share a common side, we can use the SSS postulate.



Recall that a *midline* of a triangle is a line segment joining the midpoints of two sides.

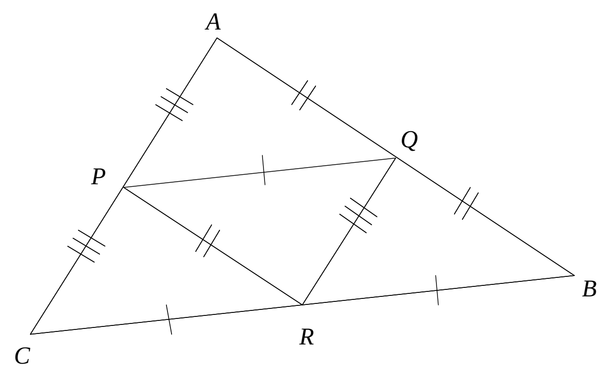
Any triangle can be divided into four congruent pieces. To do this we need the following theorem:

THEOREM ***The Midline Theorem***

A midline of a triangle is parallel to the opposite side and is half as long as the opposite side.

It is easy to use the theorem to divide any triangle into four congruent triangles.

Starting with $\triangle ABC$, let points P, Q, and R be the midpoints of the sides.



Form $\triangle PQR$. The Midline Theorem gives the following three sets of congruences:

$$\overline{PQ} \cong \overline{CR} \cong \overline{RB}$$
$$\overline{PR} \cong \overline{AQ} \cong \overline{QB}$$
$$\overline{QR} \cong \overline{PC} \cong \overline{AP}.$$

These congruences give enough information to apply the SSS postulate and conclude that

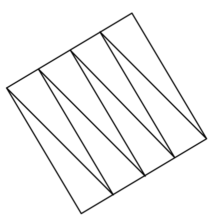
$$\triangle APQ \cong \triangle PCR \cong \triangle RQP \cong \triangle QRB.$$

Thus, triangle $\triangle ABC$ has been divided into four congruent pieces.

This problem also asks if any triangle can always be divided into two congruent pieces. This is only true for an isosceles triangle: the median from the point of intersection of the congruent sides to the third side divides the triangle into two smaller congruent triangles.

An equilateral triangle is a special case of an isosceles triangle. In an equilateral triangle, all three sides are congruent, so any median can be used to divide the triangle into two congruent triangles.

Problems 30–31 *(Student page 35)* These problems provide the opportunity for a lot of interesting explorations. Here are some hints for getting started: You can show that any triangle can be divided into four congruent pieces (see Problem 29), and then use that process to divide any triangle into $4n$ pieces, for any positive integer n. Moreover, you can try showing that any right triangle can be divided into n^2 congruent triangles (see the section "A Right Triangle Dissection" in Investigation 2.11).



It is easy to show that any square can be divided into any even number of congruent triangles. One way is to first divide the square into n congruent rectangles for any n, by dividing one side into n equal pieces, and drawing line segments to the opposite side. Then, divide each of these rectangles into two congruent triangles by a diagonal, producing $2n$ triangles. You cannot divide a square into an odd number of congruent triangles.

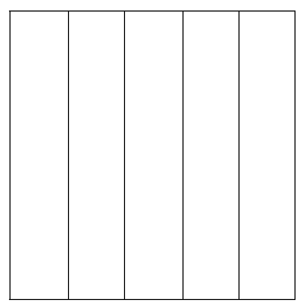
WARM-UPS FOR PROOF

Problem 1 *(Student page 36)* One approach would be show the P.E. teacher your schedule if it has a later date on it than the teacher's schedule. That would mean that yours is more recent, and hence, more accurate.

Problem 2 *(Student page 37)* If the square you start with has sides of length a, you can create four smaller squares by connecting the midpoints of opposite sides. Each new square will have sides of length $\frac{a}{2}$, and so all four squares will have the same area. Another possibility is to draw both diagonals, dividing the square into four congruent triangles. No matter how you approach the problem, you should be able to justify each step.

There are lots of ways to do this. The problem doesn't give you a particular square to start with, so you could start with 5-inch sides and mark off 1-inch lengths.

Problem 3 *(Student page 37)* The easiest way to do this is to divide the square into five congruent rectangles. If the sides of the square have length a, divide one side into five congruent pieces, each of length $\frac{a}{5}$, then draw segments from the division points, perpendicular to the opposite side. This will let you form five rectangles, each having dimensions $\frac{a}{5}$ by a.



Problem 4 *(Student page 37)*

a. Here's a sample argument to convince a fourth grade student:

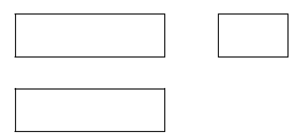
> If you start at an even number on the number line and move forward 1, 3, or 5 spaces, you will be at an odd number, because the evens and odds alternate. In fact, if you start at an even number and jump forward or backward by an odd number of spaces, you will land on an odd number. Well, when you add an odd to an even, that is exactly what you are doing—starting at an even number and moving an odd number of spaces.

WARM-UPS FOR PROOF *(continued)*

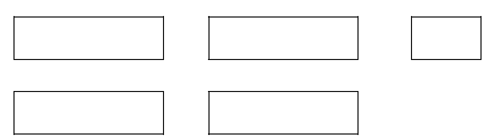
Here's a picture argument for a fourth grade student:

All even numbers look like this, because they can be divided by 2:

All odd numbers look like this, because they are one more than an even number:



Adding together an even number and an odd number, we get:

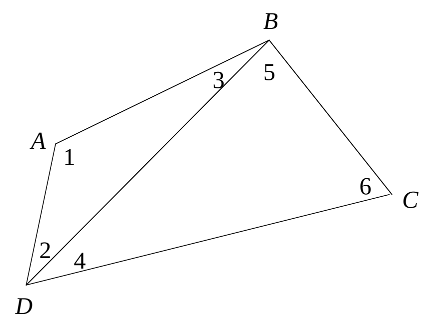


This is one more than an even number, so it must be odd.

b. Here's an argument to convince someone who knows algebra:

Any even number can be written in the form $2a$ for some integer a, while any odd number can be written as $2b + 1$ for some integer b. So, the sum of an odd number and an even number has the form $(2b + 1) + (2a)$. This simplifies to $2(a + b) + 1$, which has the form of an odd number. Therefore, when you add any odd number to any even number, you produce an odd.

Problem 5 *(Student page 37)* Take any quadrilateral and construct one of the two diagonals, forming two triangles. Suppose the angles in one triangle are labeled $\angle 1$, $\angle 2$, and $\angle 3$, while the angles in the other are labeled $\angle 4$, $\angle 5$, and $\angle 6$.



Since you know that the sum of the measures of the angles in a triangle is 180°, you know that

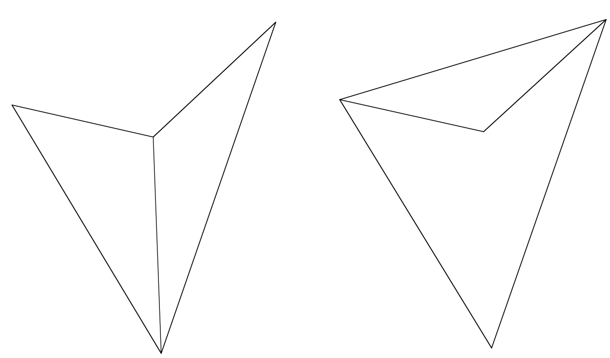
$$m\angle 1 + m\angle 2 + m\angle 3 = 180^\circ$$

and

$$m\angle 4 + m\angle 5 + m\angle 6 = 180^\circ.$$

The four angles in the original quadrilateral are $\angle A$, $\angle ABC$, $\angle C$, and $\angle CDA$. The sum of the measures of these four angles is simply the same as the sum of the measures of all six angles; by the equations above, this is equal to $180^\circ + 180^\circ = 360^\circ$.

For concave quadrilaterals, only one choice of diagonal will form two triangles from the quadrilateral, but the argument still works if you choose the appropriate diagonal.



Only the diagonal on the left forms two triangles from the original quadrilateral.

WRITING PROOFS

Problem 1 *(Student page 40)* All of the proofs use the same methods to enable them to apply the AAS postulate: the parallel lines produce two pairs of alternate interior angles, and the midpoint produces two congruent line segments. In each proof the writer carefully gives the reason for every deduction that is made.

Here are some advantages and disadvantages you might give for each proof style:

- Paragraph Proof
  - Looks more like an actual explanation as opposed to a structured mathematical device
  - Less intimidating
  - Might be difficult to read, with none of the notation displayed separately
- Two-Column Proof
  - Very organized
  - No need for lots of writing
  - The rigid structure could be intimidating
  - May be difficult to decide in what order to list the steps
- Outline Styles
  - Organized, but not as harshly as the two-column proof
  - Less writing
  - The "because" and "therefore" notation could be confusing.
  - May be harder to understand if all the details are not given.

Most of our solutions will be written as paragraph proofs.

Problem 2 *(Student page 41)* You can construct lots of examples with a ruler and protractor or with computer software. To really test the conjecture well, you want to look at a great many constructions of varying shapes and sizes. You can also use paper folding to test the conjecture. Draw two intersecting lines, then fold the vertical angles on top of each other to see if they match up.

Problem 3 *(Student page 41)* We know that $m\angle x + m\angle z = 180^\circ$ and $m\angle y + m\angle z = 180^\circ$ because each pair of angles forms a straight line. Solving for $m\angle z$ in both equations, you see that

$$m\angle z = 180^\circ - m\angle x = 180^\circ - m\angle y,$$

implying that $m\angle x = m\angle y$.

Problem 4 *(Student page 41)* The experiments performed in Problem 2 may better convince you that the conjecture is true; it's always helpful to see examples before you attempt a proof. The disadvantage of this method, however, is that no number of specific examples is ever enough to constitute a proof. No matter how many times you found the conjecture to hold, there could always be some situation in which it fails. The advantage of the proof in Problem 3 is that it tells us the statement is true for *all* pairs of vertical angles.

Problem 5 *(Student page 41)* Given any statement which begins with "for any triangle," "for all squares," or "given a parallelogram," it is impossible to perform a *complete* experiment, simply because you can't possibly look at all such objects! Sometimes you can't perform an experiment *at all*. How far away is the sun? What's the circumference of the Earth? These are just a couple of examples of questions for which you can't perform an experiment that will answer the problem directly.

The question about the Earth and the sun were solved with a combination of measurement and deduction. No one measured the Earth; people *deduced* things based on

- measurement of distance between places on the Earth;
- properties of parallel lines (the sun's rays).

If it weren't for the mathematical deductive step, the answer could not be found.

A common practice in mathematics is to take a result and see if it generalizes to other situations, such as seeing if a statement about objects in the plane holds true in three or four dimensions. Of course, once you go higher than three dimensions you can't visualize anything, so you have to rely on deduction!

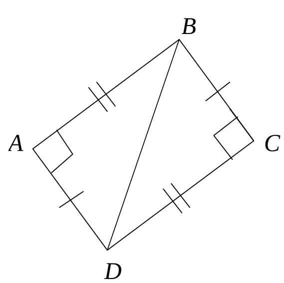
Problem 6 *(Student page 41)* This statement is true; you *can* always find two points so that the line through them misses the rest. Once again, you can investigate this conjecture by drawing examples on paper or by using geometry software. Another way to tackle it is to break it up into cases. For instance, it's given that not all five points lie on a straight line. What about the case where four of the points lie on the same line? When three points lie on the same line? This method won't let you solve the problem all at once, but it lets you break the problem into smaller, more manageable pieces. This is a good habit that is often useful in solving problems.

Problem 9 *(Student page 44)* Angles $\angle ABD$ and $\angle CBE$ are congruent, as they are vertical angles. Since it is given that $\overline{AB} \cong \overline{BC}$ and $\overline{BD} \cong \overline{BE}$, it follows that $\triangle ABD \cong \triangle CBE$ by SAS.

Problem 10 *(Student page 45)* Triangles $\triangle STV$ and $\triangle UVT$ share side $\overline{VT}$. This, combined with the given information $\overline{SV} \cong \overline{UT}$ and $\overline{ST} \cong \overline{UV}$, shows that the two triangles are congruent by SSS.

Problem 11 *(Student page 45)* Because the angles in a square are all right angles, it follows that $\angle SWB \cong \angle EBW$. Since the sides of a square all have the same length, $\overline{SW} \cong \overline{EB}$. $\triangle SWB$ and $\triangle EBW$ share side $\overline{BW}$. Therefore, by the SAS postulate, $\triangle SWB \cong \triangle EBW$.

Problem 12 *(Student page 45)* Here are two different proofs of the fact that a diagonal divides a rectangle into two congruent triangles. Notice that they are in fact different, as they appeal to different congruence postulates.

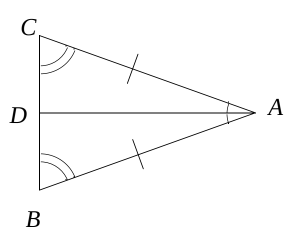


Here is the first proof: Because opposite sides of a rectangle have the same length, $\overline{AD} \cong \overline{CB}$ and $\overline{AB} \cong \overline{CD}$. Since the angles in a rectangle are all right angles, $\angle BAD \cong \angle DCB$. It follows that $\triangle ABD \cong \triangle CDB$ by SAS.

Now for the second proof. The beginning step is exactly the same, but the rest differs.

Because opposite sides of a rectangle have the same length, $\overline{AD} \cong \overline{CB}$ and $\overline{AB} \cong \overline{CD}$. It is clear that $\overline{BD}$ is congruent to itself, so it follows that $\triangle ABD \cong \triangle CDB$ by SSS.

Problem 13 *(Student page 45)* Because $\overline{XE}$ is a median, it follows that $\overline{ME} \cong \overline{YE}$. Clearly, $\overline{XE}$ is congruent to itself, and it is given that $\overline{XY} \cong \overline{XM}$. Therefore, by the SSS postulate, $\triangle XEM \cong \triangle XEY$.



Problem 14 *(Student page 45)* Suppose that $\triangle ABC$ is isosceles with vertex angle A, and let $\overline{AD}$ be the angle bisector of $\angle CAB$. By the definition of isosceles triangle, we know that $\overline{AC} \cong \overline{AB}$. Because $\overline{AD}$ is an angle bisector, $\angle CAD \cong \angle BAD$. Also, the two triangles share side $\overline{AD}$. The SAS postulate shows that $\triangle ACD \cong \triangle ABD$.

Problem 15 *(Student page 45)* It is true that $\angle ELM$ and $\angle HML$ are alternate interior angles, but this does not automatically mean they are congruent. What is true is that *if two lines are parallel, then alternate interior angles have the same measure*. In this case, you don't know that the two lines are parallel; that is what you are trying to show!

WRITING PROOFS (continued)

Problem 16 *(Student page 46)* The only mistake is the use of the ASA postulate. Two consecutive angles and a side have been shown to be congruent, so the triangles are congruent by the AAS postulate.

Problem 17 *(Student page 47)* Remember that to prove a statement is not true, you just need one counterexample. So in this case you just need two horses which are not the same color!

Problem 18 *(Student page 47)* This conjecture is not true; to see this let $n = 41$. Then

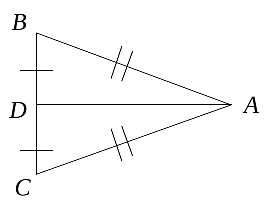
$$n^2 + n + 41 = (41)^2 + 41 + 41 = 41(41 + 1 + 1) = 41 \cdot 43.$$

So, since $41^2 + 41 + 41$ can be factored, it is not a prime number. Thus, the result is not always prime.

Analysis and Proof, Part 1

Problem 1 *(Student page 48)* One common situation in real life where you have to diagnose a situation occurs when you're looking for missing objects. For example, if you can't find your keys, it's helpful to retrace your actions in order to deduce where they must be.

Problem 2 *(Student page 49)* Since E is the midpoint of $\overline{HF}$, $\overline{HE} \cong \overline{FE}$. Because $\overline{EG}$ is perpendicular to $\overline{HF}$, it follows that $m\angle GEF = m\angle GEH = 90^\circ$, so the two angles are congruent. Side $\overline{EG}$ is shared by both $\triangle HEG$ and $\triangle FEG$, so the two triangles are congruent by SAS. Then, by CPCTC, we know that $\overline{HG} \cong \overline{FG}$.



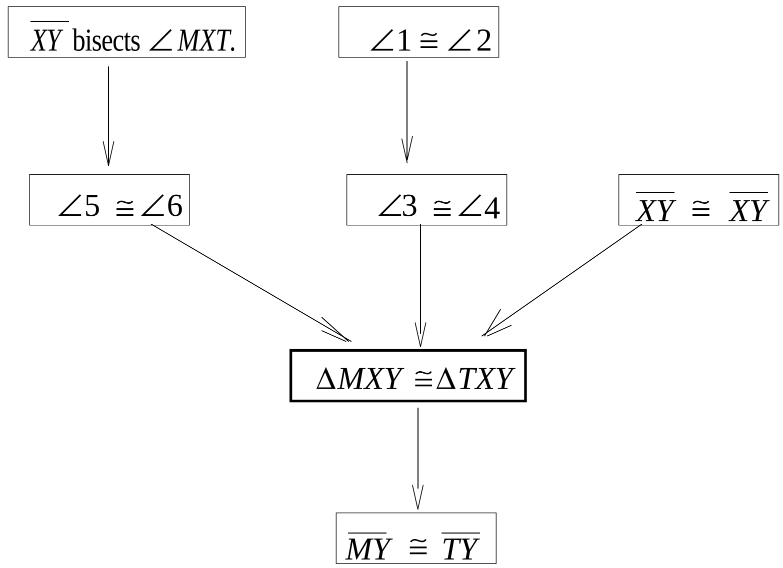
Problem 3 *(Student page 50)* Suppose triangle $\triangle ABC$ is isosceles, with $\overline{AB} \cong \overline{AC}$. Construct the median from vertex A, calling it $\overline{AD}$; this means that $\overline{DB} \cong \overline{DC}$. Triangles $\triangle ADB$ and $\triangle ADC$ share side $\overline{AD}$, and so are congruent by SSS. Then by CPCTC, it follows that $\angle B \cong \angle C$.

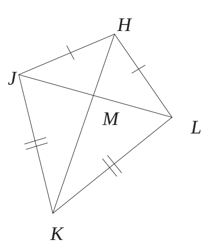
For Discussion *(Student page 50)* The CPCTC statement is a technique to be used in proofs; first show that two triangles are congruent, and then make use of that fact to show that certain corresponding parts of the two triangles are congruent. Think of CPCTC as a type of proof strategy. On the other hand, SSS, SAS, and so on, are postulates which are used to conclude that two triangles are congruent. They stand for known results—for example, if all three pairs of corresponding sides of two triangles are congruent, then the two triangles are congruent.

Problem 4 *(Student page 51)* Because $\triangle ABC$ is isosceles with $\overline{AB} \cong \overline{BC}$, you know that $\angle A \cong \angle C$. It is also given that $AE = CD$, so $\overline{AE} \cong \overline{CD}$. Therefore, $\triangle BAE \cong \triangle BCD$ by SAS. By CPCTC, it follows that $\overline{BE} \cong \overline{BD}$, implying that $\triangle BDE$ is isosceles. Then $\angle BDE \cong \angle BED$, because the base angles of an isosceles triangle are congruent (see Problem 3).

ANALYSIS AND PROOF, PART 1 *(continued)*

Problem 5 *(Student page 52)* Here is one possible flow chart:

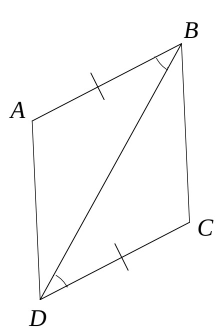




Problem 6 *(Student page 52)* Because $\overline{HJ} \cong \overline{HL}$, $\triangle JHL$ is isosceles, so $\angle HJM \cong \angle HLM$. Also, triangles $\triangle JHK$ and $\triangle LHK$ are congruent by SSS; this implies that $\angle JHM \cong \angle LHM$. Then $\triangle HJM \cong \triangle HLM$ by ASA.

Problem 7 *(Student page 53)* We will quite frequently be using the fact that a quadrilateral is a parallelogram if two opposite sides are congruent and parallel. Here is a proof:

Suppose a quadrilateral *ABCD* has one pair of opposite sides, say $\overline{AB}$ and $\overline{CD}$, parallel and the same length; we want to show that the other pair of opposite sides must also be parallel, and so *ABCD* must be a parallelogram.

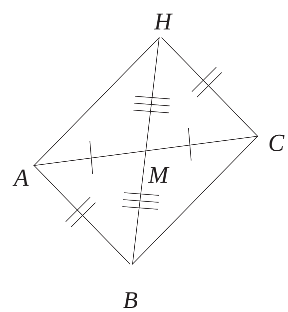


Since $\overline{AB} \parallel \overline{DC}$, it follows that $\angle ABD \cong \angle BDC$, because they are alternate interior angles. Then triangles $\triangle ABD$ and $\triangle CDB$ must be congruent by SAS. This implies that $\angle ADB \cong \angle CBD$, which means that $\overline{AD} \parallel \overline{BC}$.

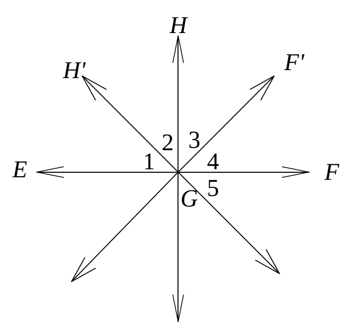
Problem 8 *(Student page 53)* The fact that $\overline{AC}$ and $\overline{BH}$ bisect each other at M does not mean that all four segments $\overline{AM}$, $\overline{CM}$, $\overline{BM}$, and $\overline{HM}$ are congruent. What it means is that $\overline{AC}$ divides $\overline{BH}$ in half, so that $\overline{BM} \cong \overline{HM}$, and also that $\overline{BH}$ divides $\overline{AC}$ in

half, giving that $\overline{AM} \cong \overline{CM}$. With this error in the visual scan, you might incorrectly conclude that *ABCH* is a rectangle.

There is, however, enough information given to conclude that *ABCH* is a parallelogram. Look at the picture below. Because the diagonals bisect each other, and it is given that $\overline{AB} \cong \overline{CH}$, you can conclude that $\triangle ABM \cong \triangle CHM$ by SSS. This implies that $\angle CHM \cong \angle ABM$, so $\overline{CH}$ is parallel to $\overline{AB}$. Since these two opposite sides are congruent and parallel, we can conclude that *ABCH* is a parallelogram.



Problem 9 *(Student page 54)* Consider the picture below.



Because $\overleftrightarrow{GF} \perp \overleftrightarrow{GH}$, it follows that

$$m\angle 3 + m\angle 4 = 90^\circ,$$

and because $\overleftrightarrow{GF'} \perp \overleftrightarrow{GH'}$, it also follows that

$$m\angle 2 + m\angle 3 = 90^\circ.$$

Putting the above together, you can see that

$$m\angle 3 + m\angle 4 = m\angle 2 + m\angle 3$$

so $m\angle 4 = m\angle 2$, implying that $\angle F'GF \cong \angle H'GH$.

Analysis and Proof, Part 1 (continued)

Problem 10 *(Student page 54)* Since *FACG* and *DABE* are squares, $\overline{FA} \cong \overline{CA}$ and $\overline{AB} \cong \overline{AD}$. Also, $\angle FAC \cong \angle DAB$ since both are 90°. Since these two angles have the same measure, we can add $m\angle BAC$ to both, obtaining equal quantities. In other words,

$$m\angle FAC + m\angle BAC = m\angle DAB + m\angle BAC,$$

but this means that $m\angle FAB = m\angle CAD$. Therefore, by SAS, $\triangle FAB \cong \triangle CAD$.

Problem 11 *(Student page 55)* Because triangles $\triangle LJM$ and $\triangle LNK$ are equilateral, $\overline{ML} \cong \overline{JL}$ and $\overline{LN} \cong \overline{LK}$. Furthermore, $\angle MLJ \cong \angle NLK$, since all angles in equilateral triangles measure 60°. You can use a trick similar to that in the solution for Problem 10, adding $m\angle JLK$ to both of these angles, yielding

$$m\angle MLJ + m\angle JLK = m\angle NLK + m\angle JLK,$$

which means that $m\angle MLK = m\angle NLJ$. Therefore, $\triangle MLK \cong \triangle JLN$ by SAS. Then you can use CPCTC to conclude that $\overline{MK} \cong \overline{JN}$.

You probably *know* that equilateral triangles have 60° angles. We haven't proved it yet, but we will soon (see Problem 7 in Investigation 2.9).

Problem 12 *(Student page 55)* Because $\triangle QRT$ is isosceles, base angles $\angle TRQ$ and $\angle TQR$ are congruent. Also, $\overline{UR} \cong \overline{VQ}$ since each has length equal to half the length of one of the two congruent sides of $\triangle QRT$. Since they share side $\overline{QR}$, triangles $\triangle URQ$ and $\triangle VQR$ are congruent by SAS. It then follows that $UQ = VR$.

Problem 13 *(Student page 55)* The fact that P is on the perpendicular bisector of $\overline{LM}$ gives two pieces of information: $m\angle PNL = m\angle PNM = 90^\circ$, and $\overline{NL} \cong \overline{NM}$. Construct segments $\overline{PL}$ and $\overline{PM}$, forming $\triangle PLN$ and $\triangle PMN$. These triangles share side $\overline{PN}$ and so are congruent by SAS. Then by CPCTC it follows that $PL = PM$.

Problem 14 *(Student page 56)* Since *QSTR* is a parallelogram, $\overline{QS} \cong \overline{RT}$, and because *STUV* is a parallelogram, $\overline{SU} \cong \overline{TV}$. (Opposite sides of a parallelogram are congruent.) So far we know that two pairs of corresponding sides of $\triangle QSU$ and $\triangle RTV$ are congruent.

Since opposite sides of a parallelogram are congruent, we also know that $\overline{QR} \cong \overline{ST}$ and $\overline{ST} \cong \overline{UV}$, implying that $\overline{QR} \cong \overline{UV}$. Moreover, since $\overline{QR}$ is parallel to $\overline{ST}$, and $\overline{ST}$ is parallel to $\overline{UV}$, it follows that $\overline{QR}$ and $\overline{UV}$ are parallel. This is enough to conclude that *QRVU* is a parallelogram, since opposite sides, $\overline{QR}$ and $\overline{UV}$, are congruent and parallel. (See the argument below.) This means that the other pair of opposite sides, $\overline{UQ}$ and $\overline{VR}$, are also congruent. Thus, all three pairs of corresponding sides in $\triangle QSU$ and $\triangle RTV$ are congruent, so the triangles are congruent by SSS.

Problem 15 *(Student page 56)* Since $\overleftrightarrow{DE}$ is parallel to $\overleftrightarrow{AC}$, we know that two pairs of alternate interior angles are congruent: $\angle DBA \cong \angle BAC$ and $\angle EBC \cong \angle BCA$.

Now, because a straight angle measures 180°, you know that

$$m\angle DBA + m\angle ABC + m\angle EBC = 180^\circ.$$

Substituting the measures of congruent angles into this equation, we have

$$m\angle BAC + m\angle ABC + m\angle BCA = 180^\circ,$$

which proves that the sum of the angles in the triangle is 180°.

Problems 16–17 *(Student pages 56–57)* The two trucks form an angle of 90° with each other at the North Pole. Each truck is driving directly to the equator, so will intersect the equator perpendicularly, forming an angle of 90°. But this means that the triangle formed will have three right angles, so the sum of the angles will be 270°! What is the problem? Well, in Problem 13 we started with the fact that, given any triangle in the plane, a line can be constructed which is parallel to one of the sides. The surface of the Earth, however, is not a plane! This means that the proof doesn't work in this new situation.

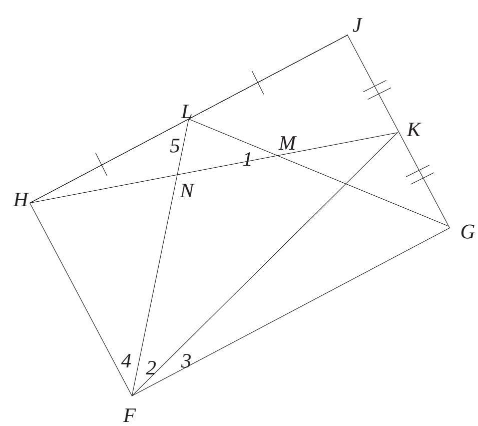
This *fact* about parallel lines is called the *parallel postulate*.

Problem 18 *(Student page 57)* The first step is to show two pairs of triangles congruent. Since L and K are midpoints, we know that $\overline{HL} \cong \overline{JL}$ and $\overline{JK} \cong \overline{GK}$. Also, because opposite sides of a rectangle are congruent, $\overline{HF} \cong \overline{JG}$ and $\overline{HJ} \cong \overline{FG}$. All angles of a rectangle are congruent. Thus, the SAS postulate gives $\triangle HLF \cong \triangle JLG$ and $\triangle JHK \cong \triangle GFK$.

This problem is difficult, as it is not at all clear how to proceed. The best approach is to just play around with various angles, writing down everything you know in an organized fashion. Eventually the correct equality will arise, as shown in the solution below.

Number angles $\angle 1$ through $\angle 5$ as in the figure below, and label point N as shown. Our goal is to show that $\angle 1 \cong \angle 2$. We will be using the following three congruences, all arising from the congruent triangles above:

$$\angle 3 \cong \angle JHK,\ \angle 4 \cong \angle JGL, \text{ and } \angle 5 \cong \angle JLG.$$



Analysis and Proof, Part 1 (continued)

Look at the line passing through H, L, and J. We see that

$$m\angle FLG = 180^\circ - m\angle 5 - m\angle JLG,$$

so $m\angle FLG = 180^\circ - 2\,m\angle 5$.

Now consider right angle $\angle JHF$. Its measure can be broken up to give

$$m\angle NHF = 90^\circ - m\angle JHK,$$

so $m\angle NHF = 90^\circ - m\angle 3$.

Adding the measures of the angles in $\triangle HNF$ yields

$$(90^\circ - m\angle 3) + m\angle 4 + m\angle HNF = 180^\circ,$$

implying that $m\angle HNF = 90^\circ + m\angle 3 - m\angle 4$.

But, using vertical angles, this shows that $m\angle LNM = 90^\circ + m\angle 3 - m\angle 4$.

Now add the measures of the angles in $\triangle LNM$ to see that

$$\begin{aligned}(90^\circ + m\angle 3 - m\angle 4) + (180^\circ - 2\,m\angle 5) + m\angle 1 &= 180^\circ \\ 90^\circ + m\angle 3 - m\angle 4 - 2\,m\angle 5 + m\angle 1 &= 0^\circ. \qquad (1)\end{aligned}$$

Looking at right angle $\angle FGK$ you see that $m\angle LGF = 90^\circ - m\angle 4$; then adding the measures of the angles in $\triangle LGF$ gives

$$\begin{aligned}(180^\circ - 2m\angle 5) + (90^\circ - m\angle 4) + (m\angle 2 + m\angle 3) &= 180^\circ \\ 90^\circ + m\angle 3 - m\angle 4 - 2m\angle 5 + m\angle 2 &= 0^\circ. \qquad (2)\end{aligned}$$

Finally, equating the two expressions and simplifying shows that $m\angle 1 = m\angle 2$, that is, $\angle LMH \cong \angle LFK$.

ANALYSIS AND PROOF, PART 2

Problem 1 *(Student page 59)* The statement that two triangles with the same area are congruent is false. There are probably people in the world with large hands but small feet, so the statement about hands and feet is probably also false, even though it is true in most cases; remember, just one counterexample is needed. The validity of the last statement probably cannot be determined!

Problem 2 *(Student page 59)*

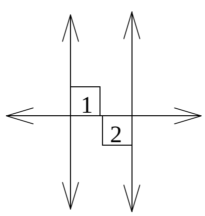
a. Hypothesis: Two lines form congruent alternate interior angles with a transversal. Conclusion: The lines are parallel.

b. Hypothesis: n is any whole number. Conclusion: $n^2 + n + 41$ is prime.

c. Hypothesis: Two sides and an included angle of one triangle are congruent to two sides and an included angle of another triangle. Conclusion: The two triangles are congruent.

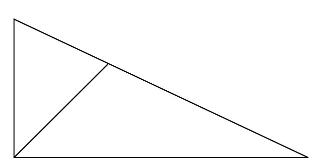
d. Hypothesis: Two lines are parallel to a third line. Conclusion: The two lines are parallel to each other.

e. Hypothesis: A shape is a rectangle. Conclusion: The area of the shape equals its length times its width.

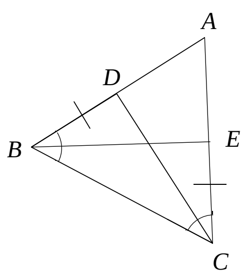


Problem 3 *(Student page 60)* The statement is true. Suppose two lines are both perpendicular to a third line (see side picture). Then angles $\angle 1$ and $\angle 2$ are congruent, as they are both right angles, but these angles are alternate interior angles formed by the third line as a transversal. Therefore, the first two lines are parallel.

Problem 4 *(Student page 60)* This statement is false. In the picture below, the angle bisector of the right angle is drawn, but this segment clearly does not bisect the opposite side of the triangle. (In other words, an angle bisector is not necessarily a median.)



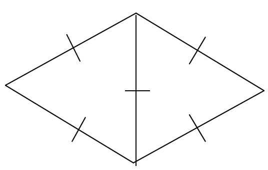
ANALYSIS AND PROOF, PART 2 (continued)



Problem 5 *(Student page 60)* The statement is true. Consider equilateral triangle $\triangle ABC$ with medians $\overline{CD}$ and $\overline{BE}$. By definition, you know that each median bisects the side it intersects; in this case all sides of the triangle are congruent, so it follows that $\overline{BD} \cong \overline{CE}$. Also, because $\triangle ABC$ is equilateral, $\angle DBC \cong \angle ECB$, as they both have a measure of 60°. $\triangle DBC$ and $\triangle ECB$ share side $\overline{BC}$ and so are congruent by SAS. This implies that $\overline{CD} \cong \overline{BE}$, so the two medians are congruent.

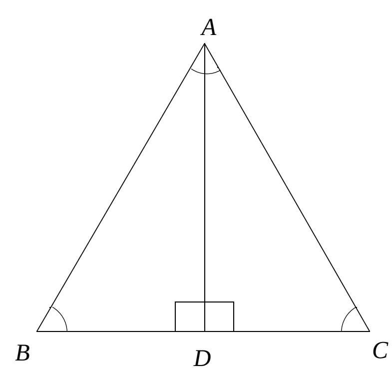
Notice that this proof will work for *any* two medians of an equilateral triangle, so the given statement is true. Also note that the above proof uses the fact that equilateral triangles are equiangular. If you haven't seen this already, a proof will be given in the next problem.

Problem 6 *(Student page 60)* The statement is true. We proved in Problem 3 of Investigation 2.8 that the base angles of an isosceles triangle are congruent. An equilateral triangle can be thought of as an isosceles triangle with *any* pair of sides serving as the two congruent ones. So the earlier proof can be applied twice in this case to two different pairs of sides, showing that all three angles of the triangle are congruent!



Problem 7 *(Student page 60)* This is false. An equilateral quadrilateral is a rhombus, and there are many examples of these that are not equiangular. For example, join two equilateral triangles to form a quadrilateral.

Problem 8 *(Student page 60)* This statement is true. Let $\triangle ABC$ be an equiangular triangle. Construct the altitude $\overline{AD}$, forming two right angles, $\angle ADB$ and $\angle ADC$. Then triangles $\triangle ADB$ and $\triangle ADC$ are congruent by AAS, implying that $\overline{AB} \cong \overline{AC}$. Repeat this same argument, this time starting with the altitude drawn from $\angle B$. This will show that $\overline{AB} \cong \overline{CB}$, and you will be able to conclude that all three sides of $\triangle ABC$ are congruent.

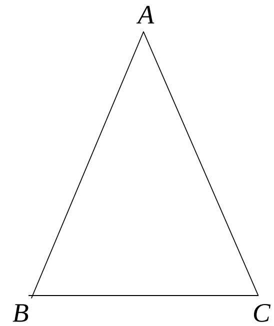


Problem 9 *(Student page 60)* This is false. Consider any rectangle which is not a square. All angles measure 90°, but the rectangle is not equilateral.

Problem 10 *(Student page 60)*

Given: $\triangle ABC$ with $\angle B \cong \angle C$.

Prove: $\overline{AB} \cong \overline{AC}$.



Look at this triangle in two ways: $\triangle ABC$ and $\triangle ACB$. We have the following corresponding congruent parts:

$$\begin{aligned} \angle A &\cong \angle A \\ \angle B &\cong \angle C \\ \overline{BC} &\cong \overline{CB}. \end{aligned}$$

Then $\triangle ABC \cong \triangle ACB$ by AAS and $\overline{AB} \cong \overline{AC}$ by CPCTC, so $\triangle ABC$ is isosceles. (Notice that we have proved that a triangle is congruent to itself, but with vertex B in one triangle corresponding to vertex C in the other.)

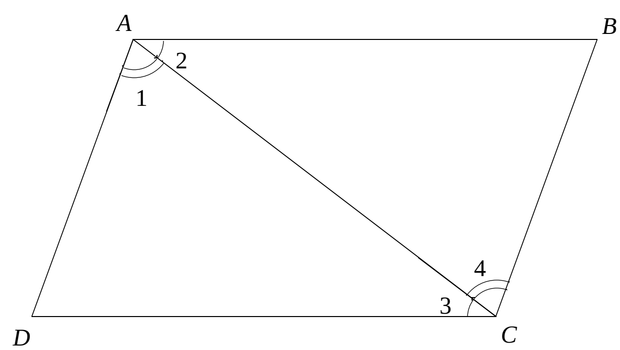
Problem 11 *(Student page 61)* Since X is the midpoint of $\overline{UT}$, you know that $\overline{TX} \cong \overline{UX}$. Because $TUVW$ is a rectangle, it follows that all angles are right angles, so $\angle U \cong \angle T$, and it also follows that opposite sides are congruent, so $\overline{TW} \cong \overline{UV}$. Therefore, triangles $\triangle WXT$ and $\triangle VXU$ are congruent by SAS. This implies that $\overline{XW} \cong \overline{XV}$, which tells us that $\triangle XWV$ is isosceles.

Problem 12 *(Student page 63)* Suppose $ABCD$ is a parallelogram. Construct the diagonal $\overline{AC}$ (see picture below). Because $\overline{AB}$ is parallel to $\overline{DC}$, $\angle 2 \cong \angle 3$, and because $\overline{AD}$ is parallel to $\overline{BC}$, $\angle 1 \cong \angle 4$, by alternate interior angles. Putting the above

statements together we see that

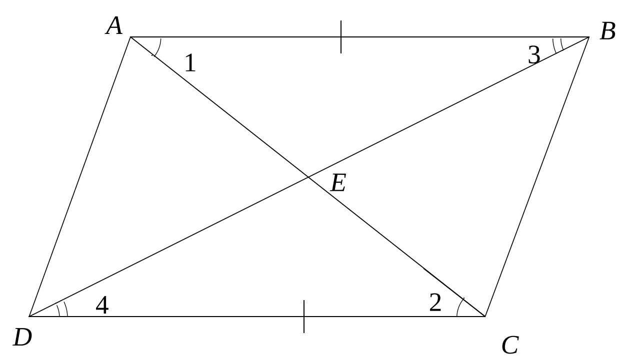
$$m\angle 1 + m\angle 2 = m\angle 3 + m\angle 4,$$

which means that $\angle A \cong \angle C$. This process can be repeated, this time drawing diagonal $\overline{DB}$, leading to the conclusion that $\angle D \cong \angle B$.



Problem 13 *(Student page 63)* Once again, suppose *ABCD* is a parallelogram and construct the diagonal $\overline{AC}$ (see picture above). As above, we can conclude that $\angle 2 \cong \angle 3$ and $\angle 1 \cong \angle 4$. This means that $\triangle ACD \cong \triangle CAB$ by ASA, since the two triangles share side $\overline{AC}$. Then by CPCTC, you can conclude that $\overline{CD} \cong \overline{AB}$ and $\overline{DA} \cong \overline{BC}$, so opposite sides of the parallelogram are congruent.

Problem 14 *(Student page 63)* In parallelogram *ABCD*, construct diagonals $\overline{AC}$ and $\overline{DB}$. Label angles 1, 2, 3, and 4 as in the picture below.



Since the opposite sides of *ABCD* are parallel, it follows that $\angle 1 \cong \angle 2$ and $\angle 3 \cong \angle 4$.

From Problem 12, $\overline{AB} \cong \overline{CD}$, so $\triangle ABE$ and $\triangle CDE$ are congruent by ASA. This implies that $\overline{AE} \cong \overline{CE}$ and $\overline{BE} \cong \overline{DE}$, so the diagonals bisect each other.

Problem 15 *(Student page 63)* Here's a different "reverse list":

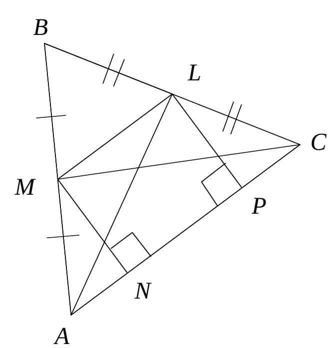
NEED: $\triangle TXB \cong \triangle ZYB$. USE: AAS

- NEED: $\angle TXB \cong \angle ZYB$.
- USE: alternate interior angles of parallel lines are congruent.
- NEED: $\angle XBT \cong \angle ZBY$.
- USE: vertical angles are congruent.
- NEED: $\overline{TB} \cong \overline{ZB}$.
- USE: diagonals of a parallelogram bisect each other.
- NEED: *TUZW* is a parallelogram.
- USE: given.

Problem 16 *(Student page 63)* $\triangle AFC$ and $\triangle AFB$ can be proved congruent using SAS. The proof is exactly like the one for Problem 13 in Investigation 2.8.

The proof below relies on the HL (hypotenuse-leg) congruence test for right triangles. This test is proved in Problem 10 of Investigation 2.11.

Problem 17 *(Student page 64)* Suppose $\overline{CM}$ and $\overline{AL}$ are congruent medians of $\triangle ABC$. Draw $\overline{ML}$. Then, $\overline{ML} \parallel \overline{AC}$ by the Midline Theorem.



***MN* = *LP* because parallel lines are everywhere equidistant.**

Now draw altitudes $\overline{MN}$ and $\overline{LP}$ to side $\overline{AC}$. Because $\overline{ML}$ and $\overline{AC}$ are parallel, it follows that $MN = LP$. This means that right $\triangle ALP$ and $\triangle CMN$ are congruent by the hypotenuse-leg theorem (this theorem is proved in Problem 10 of Investigation

Details in this proof show that if the diagonals of a trapezoid are congruent, then the trapezoid is isosceles.

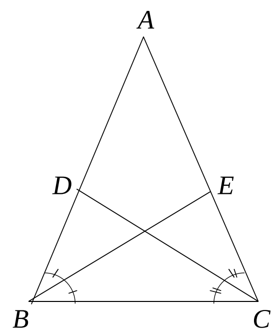
2.11). Then CPCTC implies that $\angle LAP \cong \angle MCN$. Then $\triangle ALC$ and $\triangle CMA$ are congruent by SAS, which means that $\overline{MA} \cong \overline{LC}$. Finally, this is sufficient to conclude that $\overline{BA} \cong \overline{BC}$, so $\triangle ABC$ is isosceles.

Problem 18 *(Student page 64)* This is a difficult problem. One proof is given here; another can be found in the "Mathematics Connections" section of the *Teaching Notes* for this investigation.

Given: $\triangle ABC$ with angle bisectors $\overline{BE}$ and $\overline{CD}$; $\overline{BE} \cong \overline{CD}$.

Prove: $\overline{AB} \cong \overline{AC}$.

By the definition of angle bisector, $m\angle ABE = m\angle EBC$ and $m\angle ACD = m\angle DCB$.



Compare $\angle ACB$ and $\angle ABC$. There are three possibilities:

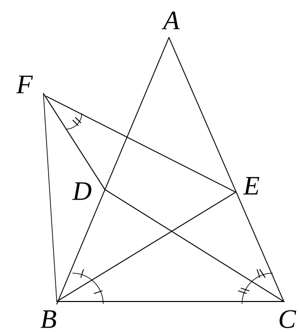
(i) $m\angle ACB > m\angle ABC$

(ii) $m\angle ACB < m\angle ABC$

(iii) $m\angle ACB = m\angle ABC$

Assume (i) $m\angle ACB > m\angle ABC$.

Construct $\overline{EF}$ such that $\overline{EF} \parallel \overline{CD}$ and $\overline{EF} \cong \overline{CD}$. Draw $\overline{FD}$ and $\overline{FB}$.



Since a pair of opposite sides are parallel and congruent, $EFDC$ is a parallelogram. Therefore, $\overline{DF} \cong \overline{EC}$.

We have $\overline{EF} \cong \overline{CD}$ and also $\overline{CD} \cong \overline{BE}$ (given), so $\overline{EF} \cong \overline{BE}$.

Thus, $\triangle FEB$ is isosceles, and $\angle EFB \cong \angle EBF$ since base angles of an isosceles triangle are congruent.

Therefore, $m\angle EFD + m\angle DFB = m\angle EBD + m\angle DBF$. Also, $m\angle EFD = m\angle ECD$, since opposite angles of a parallelogram are congruent. Since $m\angle ECD = \frac{1}{2}m\angle ACB$ and $m\angle DBE = \frac{1}{2}m\angle ABC$ (by definition of angle bisectors) and $m\angle ACB > m\angle ABC$ (by assumption), it follows that $m\angle ECD > m\angle DBE$. Thus, $m\angle EFD > m\angle DBE$. Combining this result with $m\angle EFB = m\angle EBF$ (from base angles of isosceles triangle), we see that $m\angle DBF > m\angle DFB$.

This implies that

$$DF > DB \quad (1)$$

by the following theorem:

* If two angles of a triangle are not congruent, then the sides opposite them are not congruent, and the longer side is opposite the larger angle.

Now compare $\triangle BCD$ and $\triangle CBE$. They have common side $\overline{BC}$, and $\overline{CD} \cong \overline{BE}$ (given—congruent angle bisectors). By assumption, $m\angle ECB > m\angle DBC$. Applying the Hinge Theorem to these two triangles, we conclude that

$$DB > EC. \quad (2)$$

ANALYSIS AND PROOF, PART 2 *(continued)*

** Hinge Theorem: If two sides of one triangle are congruent to two sides of a second triangle and the included angle of the first triangle is larger than the included angle of the second triangle, then the remaining side of the first triangle is longer than the remaining side of the second triangle. (The Hinge Theorem is proved in Problem 9 in Investigation 2.11.)

From inequalities (1) and (2), we have

$$DF > DB \quad \text{and} \quad DB > EC.$$

By transitivity, this implies $DF > EC$. But this contradicts the earlier statement that $\overline{DF} \cong \overline{EC}$ (or $DF = EC$) because $\overline{DF}$ and $\overline{EC}$ are opposite sides of a parallelogram.

Since our assumption (i) $m\angle ACB > m\angle ABC$ led to a contradiction, it must be false.

Now go back to the three possibilities, (i), (ii), and (iii).

We have shown that (i) is false. By symmetry, we see that (ii) is also false. (The construction and argument above can be repeated on the other side of the triangle.)

By elimination, we conclude that $m\angle ACB = m\angle ABC$, or $\angle ACB \cong \angle ABC$.

From Problem 10, we know that if two angles of a triangle are congruent, the sides opposite them are congruent. Thus, $\overline{AB} \cong \overline{AC}$. This completes the proof.

Problems 19–20 *(Student page 64)* Which method of proof you like best depends on which one coincides well with your particular strengths and work habits. The visual scan method is the quickest; you just "see"what is suppose to happen, and don't spend a lot of time writing out unnecessary details. The problem here is that all solutions might not be readily apparent. A flow chart is very well-organized; you won't be likely to omit many details. You do, however, write down a lot of information which you may ultimately not need. The reverse list combines advantages of the other two methods. It is very organized, but does not necessarily overwhelm you with facts you don't need. At each step, you decide what should be done next and how you can go about doing it.

PERSPECTIVES ON PROOF

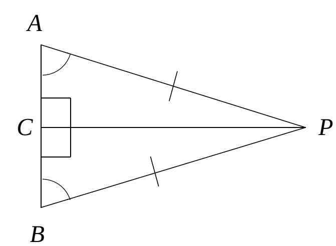
This is a reading investigation. It contains no problems, so no solutions are provided.

Perpendicular Bisectors

You proved that if a point is on the perpendicular bisector of a segment, then it is equidistant from the segment's endpoints in Problem 13 in Investigation 2.8.

You may find this proof a little subtle. You construct segment $\overline{PC}$ perpendicular to $\overline{AB}$, and then show that it also bisects $\overline{AB}$.

Problem 1 *(Student page 70)* To prove the converse, you want to prove that, if a point is equidistant from the endpoints of a line segment, then it lies on the perpendicular bisector of that segment. To do this, let P be a point which is equidistant from the endpoints of $\overline{AB}$. This means that $PA = PB$, implying that $\triangle APB$ is isosceles. Therefore, you also know that $m\angle A = m\angle B$. Now, draw a line segment from P perpendicular to $\overline{AB}$, intersecting $\overline{AB}$ at C to form $\triangle PAC$ and $\triangle PBC$. Because $\angle PCA$ and $\angle PCB$ are right angles, these two triangles are congruent by the AAS postulate. This shows that $AC = CB$, and so $\overline{PC}$ is the perpendicular bisector of $\overline{AB}$.



Problem 2 *(Student page 70)* This proof begins by showing that $\triangle SRP \cong \triangle SQP$ and $\triangle TPR \cong \triangle TPQ$. Both statements follow by SAS, using the facts that $\overline{PR} \cong \overline{PQ}$ and angles $\angle SPR$, $\angle SPQ$, $\angle TPR$, and $\angle TPQ$ are right angles. Then, using CPCTC, you can show that $\angle SRP \cong \angle SQP$ and $\angle TRP \cong \angle TQP$. It follows that

$$m\angle SRP + m\angle TRP = m\angle SQP + m\angle TQP,$$

implying that $m\angle SRT = m\angle SQT$; hence these two angles are congruent. Another method is to show that $\triangle SRT \cong \triangle SQT$ by SSS. Then, by CPCTC, $\angle SRT \cong \angle SQT$.

Problem 3 *(Student page 71)* To show that any point on $\overline{CD}$ is the same distance from A as from B, it suffices to show that $\overline{CD}$ is the perpendicular bisector of $\overline{AB}$. Let P be the point where $\overline{AB}$ and $\overline{CD}$ intersect. P will be the midpoint of $\overline{AB}$, because the fold represented by $\overline{CD}$ was obtained by folding $\overline{AB}$ in half, that is, by connecting A to B.

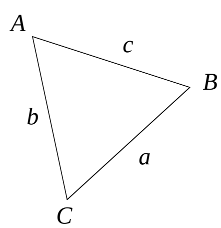
It remains to show that $\overline{CD}$ intersects $\overline{AB}$ perpendicularly. Consider the angle $\angle APB$ (where P is still the intersection point). Since this is a straight angle, its measure is 180°. The fold along $\overline{CD}$ bisects the angle; we know this because the two angles which make up the straight angle fit exactly on top of each other. Hence, they are each 90°, and so $\overline{CD} \perp \overline{AB}$.

Problem 4 *(Student page 72)*

The important idea can be summarized in the following statement:

> The perpendicular bisector of a side of a triangle passes through the opposite vertex if and only if the remaining two sides of the triangle are congruent.

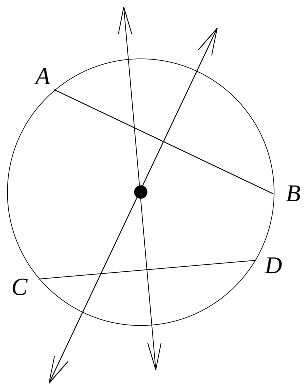
This means that if exactly one side of a triangle is to have this property, then the other two sides are congruent, so the triangle is isosceles. The triangle cannot be equilateral, however, for the above statement would then imply that all sides of the triangle would have this property.



It is not possible to have a triangle with exactly two sides having this property that their perpendicular bisectors pass through the opposite vertex. To see this, suppose you have a triangle with vertices A, B, and C, and sides a, b, c (see picture). Suppose that a and c are the two sides whose perpendicular bisectors pass through the opposite vertex. Now, use the above statement. If the perpendicular bisector of side a passes through vertex A, it follows that sides b and c have the same length. However, if the perpendicular bisector of side c passes through C, the same reasoning says that sides a and b have the same length. This means that sides c and a are congruent, which implies that the perpendicular bisector of b passes through vertex B, by the converse reasoning above. But we wanted only two sides of the triangle to have this property, not all three!

Finally, the set of triangles with all three sides having the property that their perpendicular bisectors pass through the opposite vertex is the same as the set of equilateral triangles. We saw above that if even two sides have this property, the triangle must be equilateral.

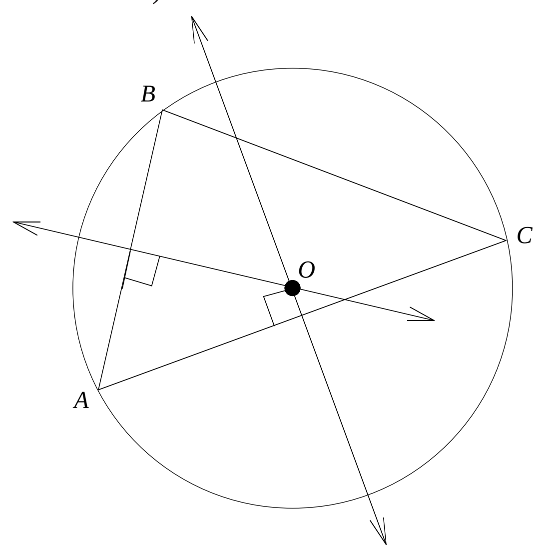
Problem 5 *(Student page 72)* To find the center of a circle, you can first construct two nonparallel chords. Then, construct the perpendicular bisectors of these chords; these two lines will intersect in one point, and this point will be the center of the circle. Here's why: Suppose the two chords are $\overline{AB}$ and $\overline{CD}$. Any point on the perpendicular bisector of $\overline{AB}$ will be equidistant from A and B, while any point on the perpendicular



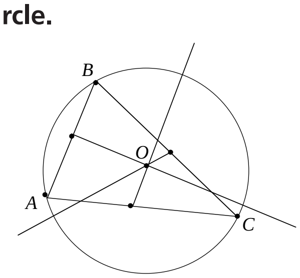
bisector of $\overline{CD}$ will be equidistant from C and D. The center of the circle is equidistant from *all* points on the circle, so it is certainly equidistant from A, B, C, and D. But any point with that property must lie on both bisectors, so it must be the intersection point of the two perpendicular bisectors. (Note that if $\overline{AB} \parallel \overline{CD}$, $\overline{AB}$ and $\overline{CD}$ will have the same perpendicular bisector, so this construction won't work.)

A polygon with a circle passing through all its vertices is called *cyclic*. All triangles are cyclic. What about other polygons?

Problem 6 *(Student page 72)* This solution to this problem uses the same reasoning as the solution for Problem 5. Suppose you want to construct a circle passing through each vertex of triangle $\triangle ABC$. You need to find the center, O, of the circle; this center will be equidistant from all three vertices. In particular, since it will be true that $OA = OB$, O must lie on the perpendicular bisector of $\overline{AB}$. Similarly, O must lie on the perpendicular bisector of $\overline{AC}$, since we need $OA = OC$. Therefore, construct these two perpendicular bisectors and find their intersection point; this point will be the center of the circle. Now, using a compass, construct the circle with center O and radius OA (or radius OB or OC).



The *circumcenter* is the center of a circumscribed circle.



Problem 7 *(Student page 72)* It is true that the perpendicular bisectors of all three sides of a triangle will meet at a point. This point is called the *circumcenter* of the triangle. Suppose that in $\triangle ABC$, point O is the point of intersection of the perpendicular bisectors of sides $\overline{AB}$ and $\overline{AC}$ (the two lines will always meet in a point). Since O lies on both perpendicular bisectors, it follows that $OA = OC$ and $OA = OB$,

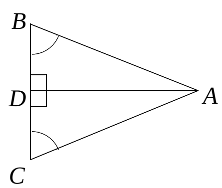
Lines which pass through the same point are called *concurrent*.

implying that $OC = OB$. This means that O is equidistant from C and B, and so it must lie on the perpendicular bisector of $\overline{BC}$. Thus, all three perpendicular bisectors intersect at the same point O. Recall that in Problem 6, this point O was used as the center of the circle passing through all three vertices.

Problem 8 *(Student page 72)* The tips of each point of the blade all lie on a circle. Using four of these points, two chords can be constructed; these chords can be used to find the center of the circle, as described in the solution for Problem 5. Or, three points can be chosen to determine a triangle. The center can be found by the intersection of the perpendicular bisectors of two sides of the triangle.

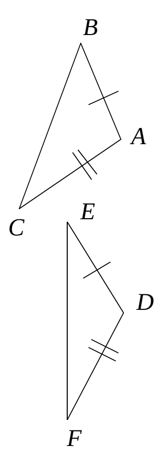
Angles and Sides in Triangles

You are really proving here that the base angles of an isosceles triangle are congruent. See the solution to Problem 3 of Investigation 2.8. The converse of this statement also holds: if a triangle has two congruent angles, then two sides must be congruent. Therefore, it is not possible for a scalene triangle to have two congruent angles.



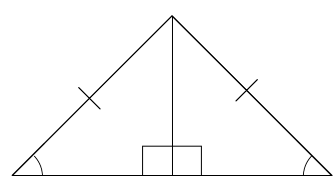
To see this, suppose you have triangle $\triangle ABC$ with $\angle B \cong \angle C$. Draw in altitude $\overline{AD}$, and use this to conclude that $\triangle ADB \cong \triangle ADC$, by AAS. Then by CPCTC, you know that $\overline{AC} \cong \overline{AB}$.

The relationship between angles and sides of a triangle is the following: the largest angle is opposite the longest side, the smallest angle is opposite the shortest side, and so the middle-size angle is opposite the middle-length side. To see this, picture each angle as a hinge. The length of the side opposite the angle is the distance between the sides of the hinge. For the largest angle, the hinge will be opened the widest, and hence will have the longest distance between its sides, and similarly for the two smaller angles.



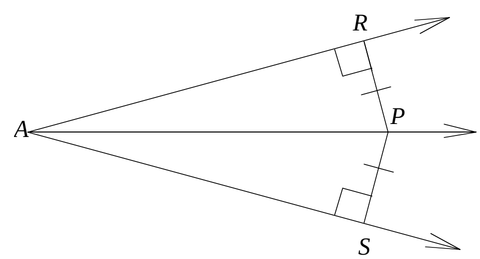
Problem 9 *(Student page 73)* Suppose that, in triangles $\triangle ABC$ and $\triangle DEF$, you know that $\overline{AB} \cong \overline{DE}$ and $\overline{AC} \cong \overline{DF}$, but $\angle A$ is larger than $\angle D$. Then side $\overline{BC}$ will be longer than side $\overline{EF}$. Picture a hinge, with fixed sidelengths AB and AC. Lie the hinge at vertex A of the first triangle, so the angle matches exactly. The distance between the two endpoints of the hinge will be the length BC. Now, if you want to lie the hinge at vertex D so that it matches up exactly, you will have to close the hinge somewhat, as $\angle D$ is smaller than $\angle A$. This forces the endpoints of the hinge closer together, which shows that the distance EF is smaller than BC.

Problem 10 *(Student page 73)* Suppose you have two right triangles with hypotenuse and one leg congruent. This means that the triangles can be positioned next to each other to form a larger triangle—the two right angles together form a straight angle, and the two congruent legs line up to form the altitude of the larger triangle.



Because the triangles have congruent hypotenuses, the large triangle is isosceles, and hence has congruent base angles. Now, the AAS postulate shows that the two right triangles are congruent.

Problem 11 *(Student page 73)* The set of all points equidistant from the sides of an angle forms the angle bisector. Suppose you have an angle with vertex A, and P is a point that is equidistant from the sides of the angle, say $PR = PS$, as in the picture below. The distance from a point to a line means the perpendicular distance. This means that $\angle PRA$ and $\angle PSA$ are right angles. If we connect A to P, we will form two congruent right triangles, $\triangle PRA$ and $\triangle PSA$; they are congruent using the "hypotenuse-leg" test from Problem 10. Using CPCTC, you can conclude that $\angle RAP \cong \angle SAP$; this means that ray PA bisects $\angle A$. Therefore, any point equidistant from the sides of the angle lies on the angle bisector.

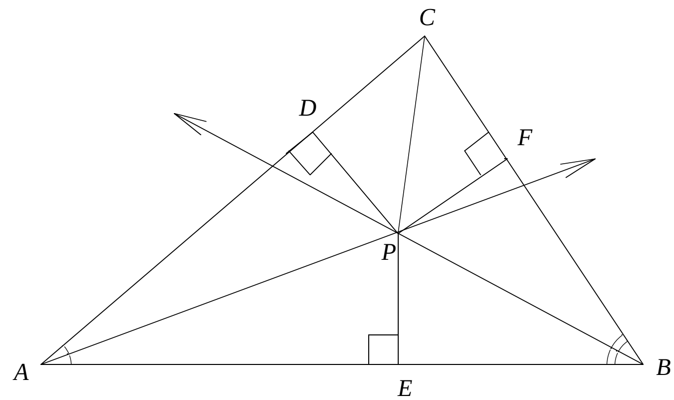


The center of the circle inscribed inside a triangle is called the triangle's ***incenter*.**

Problem 12 *(Student page 74)* The center of the circle inscribed in a triangle should be equidistant from all three sides of the triangle. This means that it should lie on the angle bisectors of all three angles, by the result of Problem 11. Therefore, to construct such a circle, construct the intersection point of the three angle bisectors, measure the distance from that point to one of the sides of the triangle (all such distances will be the same), and construct the circle with that radius.

How do you know that all three angle bisectors will intersect at a point? Two of them will, but how about the third? You can prove that the angle bisectors of a triangle are concurrent. Here's a proof:

In $\triangle ABC$, construct the angle bisectors of $\angle A$ and $\angle B$, letting P be their point of intersection. Draw perpendiculars from P to each side of the triangle, forming $\overline{PD}$, $\overline{PE}$, and $\overline{PF}$, as in the figure below.

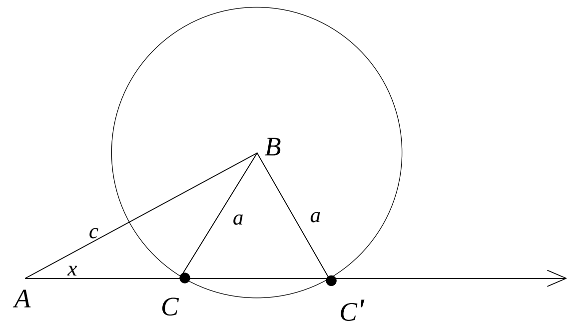


Because $\overline{AP}$ is the bisector of $\angle A$, $\overline{PD} \cong \overline{PE}$. Also, $\overline{BP}$ is the bisector of $\angle B$, so $\overline{PE} \cong \overline{PF}$. Therefore, all three of the segments perpendicular to the sides are congruent. Construct $\overline{CP}$. By the hypotenuse-leg test for right triangles, $\triangle PCD \cong \triangle PCF$, implying that $\angle PCD \cong \angle PCF$. This means that $\overline{PC}$ is the angle bisector of angle C; thus, all three angle bisectors intersect at point P.

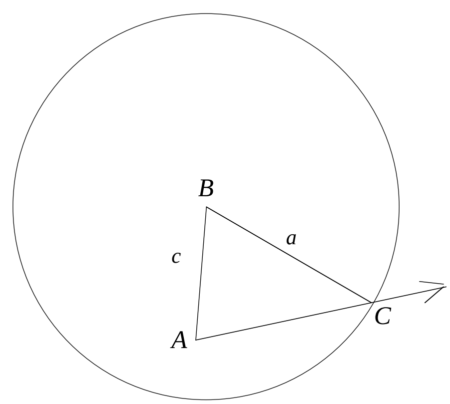
Problem 13 *(Student page 74)* Suppose you are given two sidelengths, say a and c, and the measure of $\angle A$. You want to determine whether you can find more than one triangle with two sides of length a and c, with the additional property that $\angle A$ is the non-included angle, and $\angle A$ is also the largest angle in the triangle.

In the following picture, A and B are two vertices of such a triangle. The third vertex must lie somewhere on the ray, and its distance from B must be equal to a. What you can do is to consider *all* the points in the plane which lie a distance of a away from B. This produces a circle of radius a centered at B. A circle can intersect a ray in

at most two locations, so this gives at most two possibilities for the third vertex; call them C and C'.

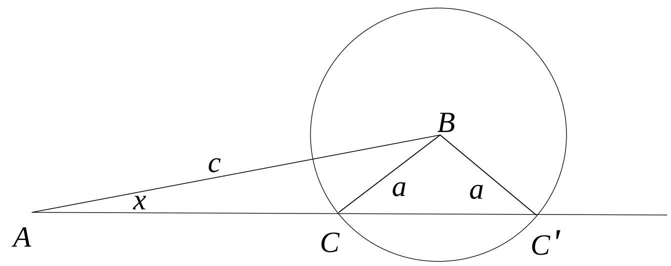


If $\angle A$ is the largest angle, then a is the longest side. Specifically, $a > c$, so the picture will look like this:



Because A is inside the circle, the ray will only intersect it in one place, producing one unique triangle. Therefore, there can be at most one triangle satisfying all of the necessary properties, so SS**A** is a congruence test.

Problem 14 *(Student page 74)* There is no SSa congruence test. Consider the picture below.



Both $\triangle ABC$ and $\triangle ABC'$ have sides of lengths c and a with a non-included angle A, where A is the smallest angle in the triangle.

Isosceles Triangle Proofs

There will be six proofs; basically each proof is centered around proving that $\triangle ACD \cong \triangle BCD$ and using CPCTC.

- Suppose statements 1 and 2 are given. Since $\triangle ABC$ is isosceles with base $\overline{AB}$, you know that $\overline{CB} \cong \overline{CA}$, and also that $\angle A \cong \angle B$ (see Problem 3 of Investigation 2.4). Further, since $\overline{CD}$ is a median, you know that $\overline{DB} \cong \overline{DA}$. By SSS, $\triangle ACD \cong \triangle BCD$, implying that $\angle ACD \cong \angle BCD$; thus $\overline{CD}$ bisects $\angle ACB$, which is statement 4. Also, by *CPCTC*, $\angle BCD \cong \angle ADC$. Since $m\angle BDC + m\angle ADC = 180^\circ$, $m\angle BDC = m\angle ADC = 90^\circ$, so it follows that $m\angle ACD = 90^\circ - x$.

  Thus, looking at the angles in $\triangle CDA$, you see that $m\angle CDA = 90^\circ$, so $\overline{CD}$ is an altitude, proving statement 3.

- Suppose statements 1 and 3 are given. The fact that $\triangle ABC$ is isosceles with base $\overline{AB}$ tells you that $\overline{CB} \cong \overline{CA}$ and $\angle A \cong \angle B$, while the fact that $\overline{CD}$ is an altitude tells you that $\angle CDA$ and $\angle CDB$ are right angles. By AAS, triangles $\triangle ACD$ and $\triangle BCD$ are congruent, implying that $\overline{DB} \cong \overline{DA}$ and $\angle ACD \cong \angle BCD$; these statements show that $\overline{CD}$ is a median (statement 2) and $\overline{CD}$ bisects $\angle ACB$ (statement 4).

- Now suppose statements 1 and 4 are given. Together they tell you that $\overline{CB} \cong \overline{CA}$, $\angle A \cong \angle B$, and $\angle ACD \cong \angle BCD$. Then ASA lets you conclude that $\triangle ACD \cong \triangle BCD$. This shows that $\overline{CD}$ is a median, as $\overline{DB} \cong \overline{DA}$ (statement 2). Repeating the argument in the first proof lets you conclude that $\overline{CD}$ is an altitude, proving statement 3.

- This time assume statements 2 and 3. Therefore, $\overline{DB} \cong \overline{DA}$ and $\angle CDA$ and $\angle CDB$ are right angles. Now you know that $\triangle ACD \cong \triangle BCD$ by SAS, which lets you conclude statements 1 and 4.

- Assume statements 2 and 4. This case may be the most difficult. It turns out that you need to use the following theorem:

THEOREM ***Angle Bisector Theorem***

An angle bisector in a triangle divides the opposite side into segments that have the same ratio as the other two sides. In other words, if $\overline{CD}$ bisects $\angle ACB$ in $\triangle ABC$, then $\frac{AD}{DB} = \frac{CA}{CB}$.

So, in this situation, we apply this theorem to see that statement 4 implies that $\frac{AD}{DB} = \frac{CA}{CB}$. But because $\overline{CD}$ is a median, you know that $\frac{AD}{DB}$ is equal to 1. Therefore,

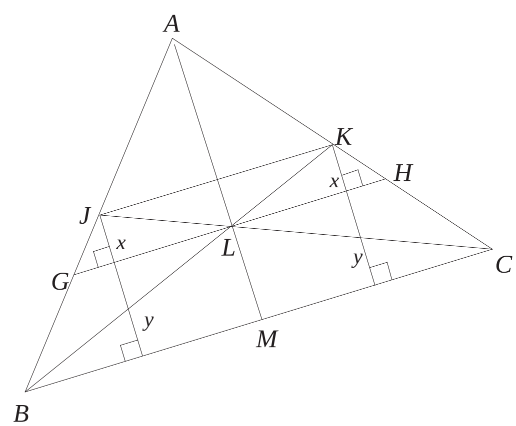
it follows that $CA = CB$, so the triangle is isosceles. Now, you know statements 1, 2, and 4, so it is possible to prove statement 3.

- Finally, assume statements 3 and 4. You thus know that $\angle CDA$ and $\angle CDB$ are right angles, and $\angle ACD \cong \angle BCD$. This time it is ASA which lets you conclude that $\triangle ACD \cong \triangle BCD$, which is enough to prove statements 1 and 2.

Problem 15 *(Student page 75)* The statement "triangles *ACD* and *BCD* are congruent right triangles" guarantees that all four given statements are correct. Because the two right triangles are congruent, you know that $\overline{AD} \cong \overline{BD}$, so in $\triangle ABC$, $\overline{CD}$ is a median. You also know that $\overline{AC} \cong \overline{BC}$, so $\triangle ABC$ is isosceles, and $\angle ACD \cong \angle BCD$, so $\overline{CD}$ is an angle bisector. Finally, since $\angle ADC$ and $\angle CDB$ are right angles, it follows that $\overline{CD}$ is an altitude of $\triangle ABC$.

In fact, "triangles *ACD* and *BCD* are congruent" is enough. How can you conclude from this and the picture that they are right triangles?

Problem 16 *(Student page 76)* This problem is easiest to do if you assume a knowledge of similar triangles. In $\triangle ABC$, let J and K be the midpoints of sides $\overline{AB}$ and $\overline{AC}$, respectively. Let L be the point of intersection of $\overline{CJ}$ and $\overline{BK}$, and draw segment $\overline{GH}$ through L, parallel to $\overline{BC}$. One of the key facts we will be using is that all three segments $\overline{JK}$, $\overline{GH}$, and $\overline{BC}$ are parallel (notice that $\overline{JK} \parallel \overline{BC}$ by the Midline Theorem). Now, the height (perpendicular distance) from J to $\overline{GL}$ is the same as the height from K to $\overline{HL}$; call this height x. Similarly, the height from J to $\overline{BC}$ is equal to the height from K to $\overline{BC}$; call this height y.



The goal is to prove that the line through A and L bisects $\overline{BC}$. Let M be the point of intersection of this line with $\overline{BC}$.

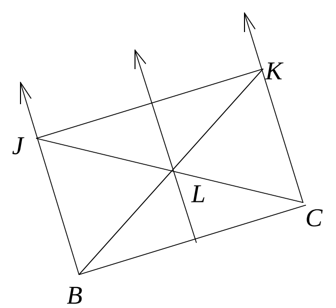
First of all, notice that $\triangle JGL \sim \triangle JBC$ and $\triangle KHL \sim \triangle KCB$, since $\overline{GH} \parallel \overline{BC}$. Because of this similarity, you know that $\frac{x}{y} = \frac{GL}{BC}$ and $\frac{x}{y} = \frac{LH}{BC}$, implying that $\frac{GL}{BC} = \frac{LH}{BC}$.

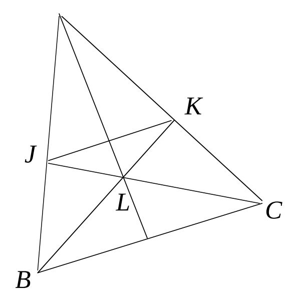
This means that $\overline{GL} \cong \overline{LH}$. So L is the midpoint of $\overline{GH}$.

Now, because $\overline{GH} \parallel \overline{BC}$, it also follows that $\triangle ABM \sim \triangle AGL$ and $\triangle ACM \sim \triangle AHL$. Therefore, it follows that $\frac{AL}{GL} = \frac{AM}{BM}$ and $\frac{AL}{LH} = \frac{AM}{CM}$. Since $GL = LH$, however, you see that $\frac{AM}{BM} = \frac{AM}{CM}$. This implies that $\overline{BM} \cong \overline{CM}$, so M is the midpoint of $\overline{BC}$.

Notice that in solving this problem, you never used the fact that $\triangle ABC$ is isosceles; you just used that J and K are midpoints, enabling you to apply the Midline Theorem.

If you are familiar with perspective drawings, there is another way to visualize this problem. Draw a rectangle $JKCB$, and its diagonals, $\overline{JC}$ and $\overline{BK}$. The line connecting the midpoints of opposite sides of the rectangle passes through L, the intersection of the diagonals, and is parallel to the other two sides of the rectangle.





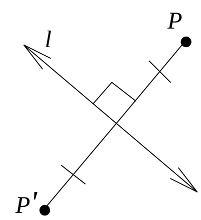
Now make a perspective drawing of the same situation. Because concurrence of points does not change in perspective, the line through the midpoints of opposite sides still passes through L. The three parallel lines (the line passing through L and the two sides of the rectangle) now converge at the vanishing point, which can be thought of as the vertex of $\triangle ABC$ at A. Thus, the line connecting A to base $\overline{BC}$ of the triangle and passing through L bisects $\overline{BC}$.

Problem 17 *(Student page 76)* Suppose you start with the assumptions that $\overline{WP}$ and $\overline{KD}$ are parallel, and $\overline{WK}$ and $\overline{PD}$ are also parallel. This makes $WPDK$ a parallelogram, and from this all the other statements can be concluded. If you say that $KWPD$ is a parallelogram, this one statement is enough to prove all six of those given, and therefore any two which together imply that $KWPD$ is a parallelogram will suffice (such as $\overline{WK} \parallel \overline{PD}$ and $WK = PD$).

A Reflection Puzzle

It helps to think of *l* as a mirror.

The first step is to have a good definition of a reflection. Suppose you want to reflect a point P about a line l. If P is on l, then it remains fixed under the reflection. If P is not on l, then P gets sent to the point P' with the property that l is the perpendicular bisector of $\overline{PP'}$. So to perform the reflection, you draw a line through P perpendicular to l, and choose the point P' on this line which lies on the opposite side of l and at the same distance from l as is P.



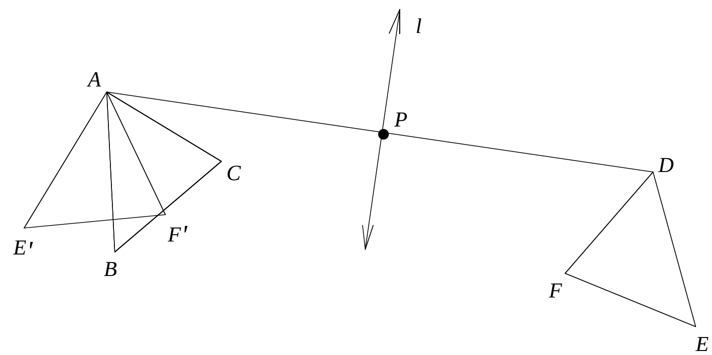
To reflect a triangle about a line, you can reflect each vertex and connect the new vertices to obtain the reflected triangle. This is because reflections preserve distance.

The goal here is to discover that if two distinct congruent triangles lie in the plane, then the minimum number of reflections needed to map one to the other is 1, while the maximum number is 3.

Problem 18 *(Student page 77)* Suppose $\triangle ABC$ and $\triangle DEF$ are congruent. The directions below will map $\triangle DEF$ onto $\triangle ABC$ in three steps. Notice however, that if the triangles are positioned in certain ways, the process may be completed in fewer than three steps; in that case, the remaining steps will have no effect.

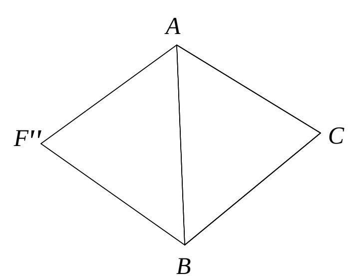
Step 1: Pick two corresponding vertices, one from each triangle, say A and D. Form segment $\overline{AD}$, let P be the midpoint of $\overline{AD}$, and let line l be the perpendicular bisector of $\overline{AD}$, noticing that it passes through P. Now, reflect $\triangle DEF$ about l. Notice that D

goes to A, by the definition of reflection. Name the images of E and F as E' and F', respectively.



Step 2: You now have congruent triangles $\triangle ABC \cong \triangle DEF \cong \triangle AE'F'$. Pick two new corresponding vertices, one from the original triangle, and one from the new reflected triangle, say B and E'. Construct $\overline{BE'}$, let Q be its midpoint, and let m be the perpendicular bisector of $\overline{BE'}$. Reflect $\triangle AE'F'$ about m.

By definition of reflection, E' goes to B. It also is true, however, that A remains at A. This could only happen if A is on line m. But notice that because reflection preserves distance, $AE' = AB$ (as they are corresponding parts of congruent triangles). This means that A is equidistant from the endpoints of $\overline{BE'}$, which means that A lies on the perpendicular bisector of $\overline{BE'}$ (see Problem 13 in Investigation 2.8). Call the image of F' under the reflection F''. So the image of $\triangle AE'F'$ under the reflection is $\triangle ABF''$, which is congruent to $\triangle ABC$, and shares side $\overline{AB}$.

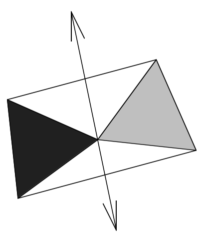


Step 3: One more reflection: Construct $\overline{CF''}$; its perpendicular bisector will be $\overline{AB}$. Because $\triangle ABF'' \cong \triangle ABC$, you know that $\overline{AF''} \cong \overline{AC}$ and $\overline{BF''} \cong \overline{BC}$. This means

that both A and B are equidistant from the endpoints of $\overline{CF''}$, so both must lie on the perpendicular bisector, implying that $\overline{AB}$ is the perpendicular bisector.

Reflect $\triangle ABF''$ about $\overline{AB}$. The reflection fixes A and B, while sending F'' to C. Therefore, the image is $\triangle ABC$, completing the process.

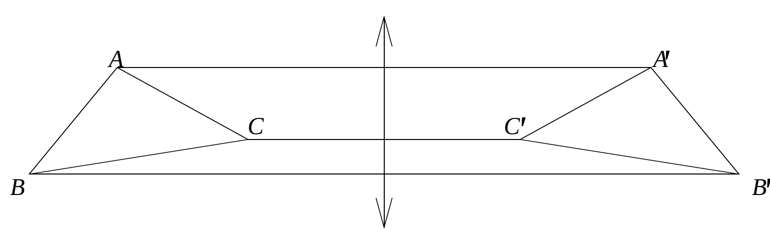
Problem 19 *(Student page 77)* If the two triangles you start with are equilateral, the process above should be done in two steps. In other words, the image under the second reflection is the original triangle, so there is no need for the final step.



To see this, picture what happens after the first reflection; you have two congruent equilateral triangles which share a vertex. The remaining four vertices form an isosceles trapezoid. You can connect the midpoints of the two parallel sides, and the line passing through these midpoints will also pass through the common vertex. Now, reflect about this line, and the triangles will agree.

Problem 20 *(Student page 77)* If one triangle is a reflection of the other about some line, then it certainly takes only one reflection to map one to the other.

The two triangles below can also be mapped one on top of the other using only one reflection. Notice a special property of these two triangles which makes this possible: if you connect all three pairs of corresponding vertices, you get three parallel line segments.

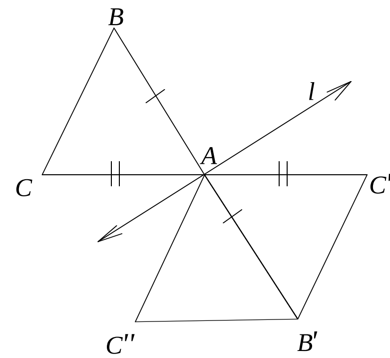


In fact, one way to define a rotation is as a composition of two reflections.

Problem 21 *(Student page 77)* If a triangle is rotated 180° about a point, it can be mapped to its image using two reflections. To demonstrate this, we shall look at the case where the triangle is rotated about one of its vertices, as this is the easiest to analyze.

Suppose $\triangle ABC$ is rotated 180° about A; call its image $\triangle AB'C'$. Let line l be the perpendicular bisector of $\overline{BB'}$, noticing that it passes through A ($AB = AB'$, as rotation preserves distance). Reflect $\triangle ABC$ about l. Notice that A is fixed, and the image of

B is B'; let C'' be the image of C, so the image of $\triangle ABC$ is $\triangle AB'C''$. Now reflect $\triangle AB'C''$ about $\overline{AB'}$. The image will be $\triangle AB'C''$, so you are finished.



If the point P is not one of the vertices of the triangle, it is still possible to map one triangle to the other with two reflections. One way to do this is to take any two lines which intersect perpendicularly at P; reflecting about these two lines will do the trick.

Problem 22 *(Student page 77)* In this problem, the line of reflection has the equation $y = x$. This makes sense, since you are merely exchanging the x- and y-coordinates in two vertices of the triangle, and the third vertex is fixed at the origin.

Problem 23 *(Student page 77)*

a. The new coordinates are

$$X' = (0, 0),\ \ Y' = (-3, 4),\ \ Z' = (-2, 5).$$

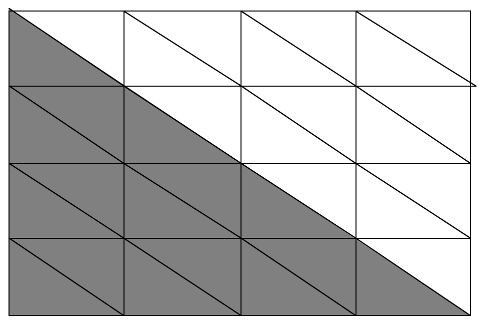
b. In fact, if $P = (a, b)$ is any point in the plane, the image of P under a 90° counterclockwise rotation is the point $P' = (-b, a)$. You can show that the two lines connecting each point to the origin are perpendicular.

Moreover, reflecting about the lines $y = x$ and $y = 0$ (or any other two lines which intersect at a 45° angle) will have the same effect as rotating 90° counterclockwise.

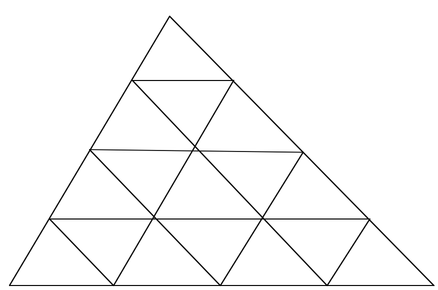
A Right Triangle Dissection

Suppose you start with a right triangle with legs of length a and b, and you want to dissect it into n^2 congruent triangles. Here's one way to do it. First, form the rectangle with dimensions $a \times b$. The rectangle can be easily divided into n^2 smaller rectangles, by dividing each side into n pieces and connecting opposite points. There will be n^2 smaller rectangles, each with dimensions $\frac{a}{n} \times \frac{b}{n}$. If one of the diagonals of each of these smaller rectangles is drawn in, you will have $2n^2$ congruent triangles. Now, split the larger rectangle into two pieces by one of its diagonals; each one of the two pieces will be congruent to the original right triangle. Since the whole rectangle contained $2n^2$ congruent triangles, the original right triangle will contain n^2 congruent triangles, as it is one half of the rectangle.

Below is a picture for the case where $n = 4$.



Problem 24 *(Student page 78)* In the above construction, we used the fact that two right triangles together form a rectangle; this is not true for arbitrary triangles, so a different method would have to be used. In fact, it is possible using an extension of the Midline Theorem.



Divide each side of an arbitrary triangle into n congruent segments, then connect

division points, always drawing parallel to the triangle's sides. You will get n^2 smaller congruent triangles.

Problem 25 *(Student page 78)* The original triangle is divided into n^2 smaller ones, so each smaller triangle has area equal to $\frac{1}{n^2}$ times the area of the original.

Problem 26 *(Student page 78)* Each smaller triangle will have perimeter that is $\frac{1}{n}$ times the perimeter of the larger triangle. This is because each side of the smaller triangles is $\frac{1}{n}$ as long as the corresponding side of the original triangle.

Connecting Midpoints on a Tetrahedron

Each corner which is cut off of the tetrahedron will be a smaller tetrahedron. The remaining solid will be an octahedron: it will have 4 faces which are the remains of the 4 sides of the original tetrahedron, and it will have 4 new faces, which are formed when each corner is cut off. Each face of this octahedron will be a triangle.

If the midpoints of a cube are connected, and the corners cut off, the shape which is cut off from each corner is still a tetrahedron (four triangular faces), but is not a *regular* tetrahedron. If the original square has sides of length 1, the base of the tetrahedron will be an equilateral triangle with sides of length $\frac{\sqrt{2}}{2}$, and it will have 3 faces, each a right triangle with sides of length $\frac{1}{2}$, $\frac{1}{2}$, and $\frac{\sqrt{2}}{2}$. What is left of each original face of the cube is a square with sides of length $\frac{\sqrt{2}}{2}$, and the new solid will have 8 new faces (one from each corner cut off); each new face will be an equilateral triangle with sides of length $\frac{\sqrt{2}}{2}$ (congruent to the bases of the pyramids which are cut off).

Problem 27 *(Student page 79)* This game can be played with other solids; you are encouraged to investigate on your own. In general, after cutting off corners, the faces of the resulting solid will depend on how many faces meet at each corner of the original solid. If there are four faces meeting at each vertex (as in an octahedron), the faces formed by the cutoff will be quadrilaterals (and squares if it's a regular polyhedron). The shape that is cut off will have one more face than the number that meet at each vertex of the original. In general, they will be pyramids (triangular faces) with differing bases.

If you start with a triangle with sides of length *a*, then all answers here will be multiplied by *a*.

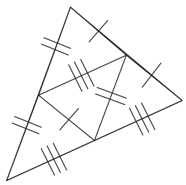
Problem 28 *(Student page 80)* For the spiral, suppose you start with an equilateral triangle with sides of length 1. The first nested triangle will be equilateral with sides of length $\frac{1}{2}$, the second with sides of length $\frac{1}{4}$, the third with sides of length $\frac{1}{8}$, and so on. If the spiral has 5 arms, it consists of one side from each of the first 5 triangles. Thus its length will be $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}$, which is equal to $\frac{16+8+4+2+1}{32} = \frac{31}{32}$. The length will always be $\frac{2^n-1}{2^n}$, where n is the number of arms.

Problem 29 *(Student page 80)* Noticing the pattern above, a spiral with 10 arms will have a length of $\frac{2^{10}-1}{2^{10}}$, and one with 100 arms will have a length of $\frac{2^{100}-1}{2^{100}}$.

As *n* gets very large, the number $\frac{1}{2^n}$ gets very small.

Problem 30 *(Student page 80)* As the number of arms gets bigger, the length of the spiral gets closer and closer to 1 (or to a, if that's the sidelength of the first triangle). In other language, the limit of the length of the spiral as the number of arms go to infinity is 1.

Congruence from Parts



If two triangles have congruent midlines (segments which connect the midpoints of the sides), then the two triangles are congruent. To see this, you once again need the Midline Theorem, which states that any midline of a triangle is half the length of the opposite side. Therefore, if you are given all three midlines, you also know all three sides, but we know that this completely determines the triangle (SSS). Because the lengths of the three midlines determine the triangle uniquely, they form a suitable test for triangle congruence.

It is also true that if two triangles share the same length altitudes, they are congruent. Suppose $\triangle ABC$ and $\triangle A'B'C'$ have altitudes of length h_a, h_b, and h_c, where h_a is the length of both the altitude from A and the altitude from A', and so on. Notice that this already fixes a correspondence between vertices and sides of the two triangles.

Now, using that the area of a triangle equals one half base times height, you can write three different equations for the area of each triangle, and then solve for each side-length in these equations. So, if the area of $\triangle ABC$ is K, while the area of $\triangle A'B'C'$ is K', you get that

$$AC = \frac{2K}{h_b} \qquad BC = \frac{2K}{h_a} \qquad AB = \frac{2K}{h_c}$$

and also that

$$A'C' = \frac{2K'}{h_b} \qquad B'C' = \frac{2K'}{h_a} \qquad A'B' = \frac{2K'}{h_c}.$$

Now use the above equations to calculate the ratios of corresponding sides, yielding

$$\frac{AC}{A'C'} = \frac{K}{K'} \qquad \frac{BC}{B'C'} = \frac{K}{K'} \qquad \frac{AB}{A'B'} = \frac{K}{K'},$$

so it follows that

$$\frac{AC}{A'C'} = \frac{BC}{B'C'} = \frac{AB}{A'B'}.$$

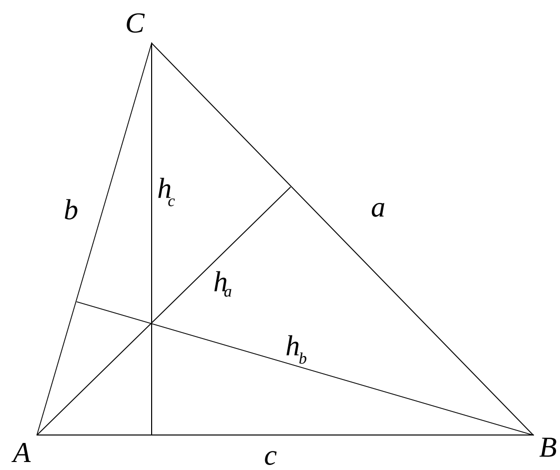
Two sets of corresponding lengths are *in proportion* if the ratios of corresponding pairs are equal.

This means that the sides of the two triangles are proportional, implying that one triangle is a scaled copy of the other, meaning that one is just an enlargement of the other. But remember that the two triangles share the same altitudes, so if one were really a shrunken version of the other, then its altitudes would be shorter. Therefore, the triangles must really be congruent. In other words, they are scaled copies of each other, but scaled by a factor of 1!

Problem 31 *(Student page 81)* This problem amounts to turning the proof that three altitudes determine a triangle into a construction. You also need to be given a unit length.

INVESTIGATIONS AND DEMONSTRATIONS
(continued)

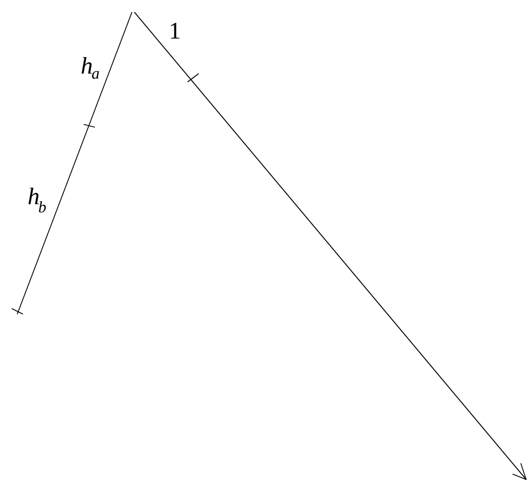
As mentioned in the notes above, the three altitudes determine the ratios of the three sides:



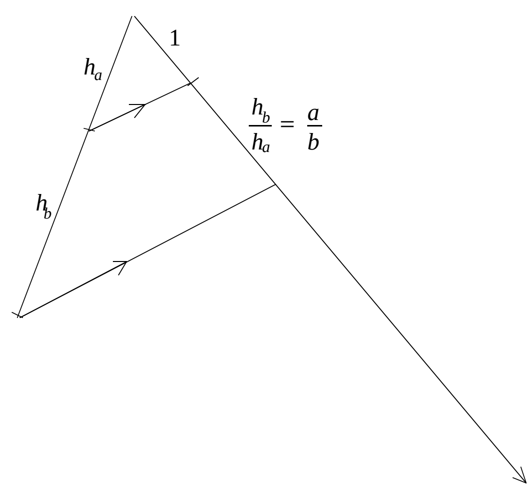
Since $ah_a = bh_b$, we have $\frac{a}{b} = \frac{h_b}{h_a}$. And it's the same for the other sides. Hence we know $\frac{a}{b}$ and $\frac{c}{b}$. Construct these lengths. Then construct a triangle whose sides have length $\frac{a}{b}$, $\frac{c}{b}$, and 1. Now dilate this triangle by a factor of $\frac{h_b}{H_b}$, where H_b is the height of this newly constructed triangle corresponding to h_b (any other altitude would do as well), and you have the desired triangle.

It remains to show how to construct $\frac{a}{b}$ and $\frac{c}{b}$ and how to dilate a triangle by a given magnitude.

If you know 1, h_a, and h_b, lay off these lengths as marked:

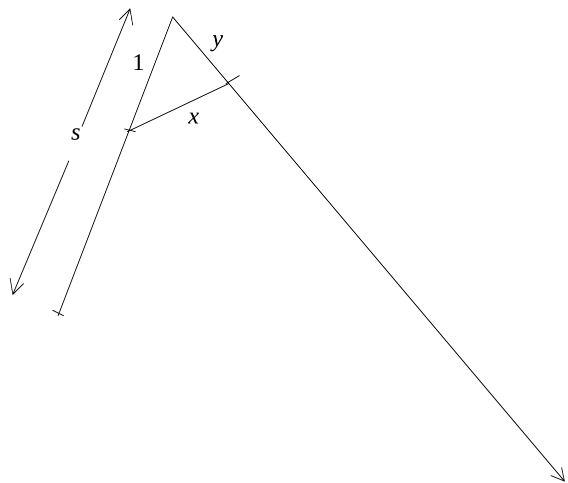


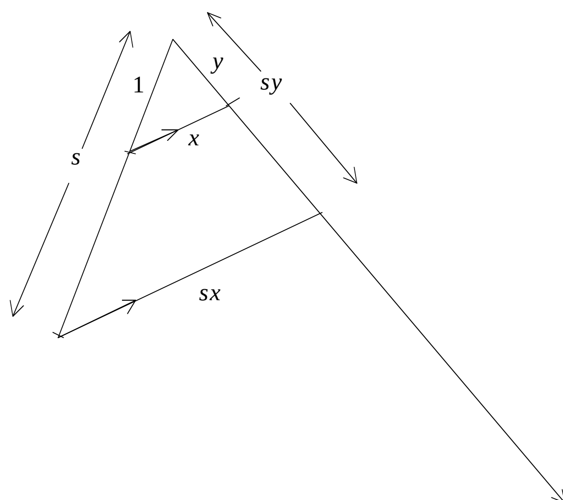
Lay off h_a, and h_b on a segment, make a ray (at any angle with the segment) and mark off 1 (the unit length). Now join the end of "h_a" to the end of "1", and from the end of h_b, construct a parallel to this "midline:"



This cuts off a segment of length $\frac{a}{b}$.

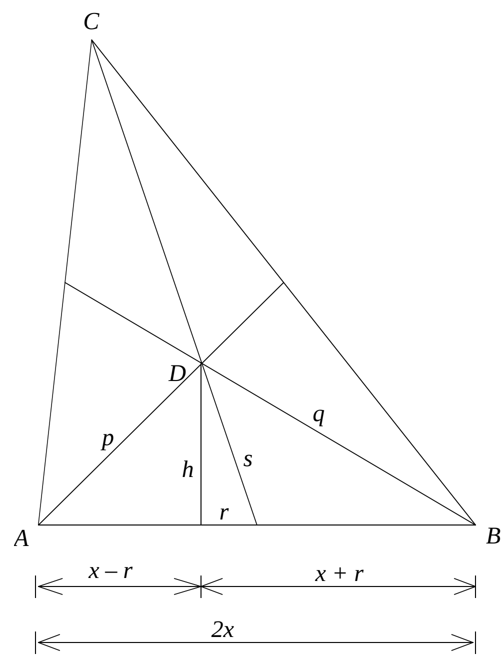
As for scaling by s, suppose you have a triangle whose sidelengths are 1, x, and y. Lay out a segment of length s along the side of length 1, and the use the "parallel" method:





Construct a parallel to the segment of length x.

Problem 32 *(Student page 81)* Look at the following picture:



The *centroid* of a triangle is the point of intersection of the three medians. A fact which you may have already conjectured is that the distance from each vertex to the centroid is two thirds the length of the corresponding median.

Here's a triangle with three medians. We want to show that the triangle is determined by the lengths of the medians. This amounts to showing that the sidelengths are determined by the medians, since SSS completely determines a triangle. Let $2x$ be the length of side $\overline{AB}$ (so the median from C divides $\overline{AB}$ into two pieces of length x). By symmetry, it's enough to show that $2x$ can be expressed in terms of the medians. Now, D is the centroid, so p and q are each $\frac{2}{3}$ of the length of a median, and s is $\frac{1}{3}$ of the length of the remaining median. Thus, p, q, and s are determined by the medians, and we'll be done if we write x in terms of p, q, and s. Drop a perpendicular from D, and label the legs h and r as shown.

Then

$$(x-r)^2 + h^2 = p^2$$
$$(x+r)^2 + h^2 = q^2$$
$$r^2 + h^2 = s^2.$$

Replacing $r^2 + h^2$ by s^2 in the first two equations and then adding, we can solve for x^2 (and hence x and $2x$) in terms of p, q, and s. Now repeat this process, solving for the remaining two sides of the triangle.

Because we're able to solve for x, which determines the base length in $\triangle ABD$ in terms of p, q, and s (the sides and a median of $\triangle ABD$) we have shown that we can determine a triangle with two sides and the included median (an SMS theorem).

Problem 33 *(Student page 81)* This is a very difficult problem, originally dating back to 1875. For a discussion of the problem and solution, see "*The Existence of a Triangle with Prescribed Angle Bisector Lengths*" by Petru Mironescu and Laurentiu Panaitopol, in *The American Mathematical Monthly*, 101, no. 1 (January 1994), pp. 58–60. The idea behind the proof is that, using some trigonometry, you can relate the length of each angle bisector to the three lengths of the sides of the triangle. You get three equations in three unknowns, which you can solve for the lengths of the sides and have SSS congruence.

Making Quadrilaterals from Congruent Triangles

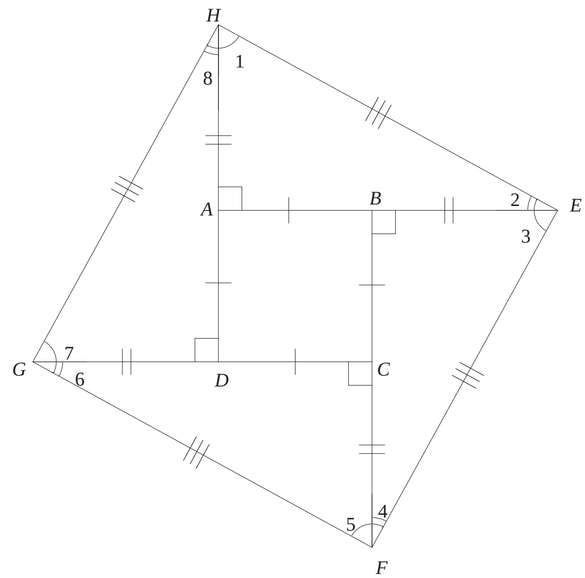
For each pair of triangles, there will be three all-red figures and three mixed-color figures that you can form. Each one is obtained by lining up corresponding sides of the two triangles, with the appropriate sides facing up. Notice that if you don't line up the sides of the same length, you will not obtain a quadrilateral. We list below all the possible outcomes.

- The 3″, 4″, 6″ triangles:
  - All three of the all-red quadrilaterals obtained are parallelograms.
  - The mixed-color quadrilateral formed by adjoining the two sides of length 6 is a standard kite; it has two consecutive sides of length 3″, and two consecutive sides of length 4″. The two mixed-color quadrilaterals formed by adjoining the sides of length 3″ or 4″ are technically kites (two pairs of consecutive sides congruent), but have a bit of a degenerate shape.
- The 3″, 4″, 5″ right triangles:
  - Again, all three of the all-red quadrilaterals are parallelograms, although the figure obtained by adjoining the sides of length 5″ is a rectangle.
  - Two of the mixed-color figures are not quadrilaterals at all—they are triangles! These are the two formed by lining up the sides of length 3″ and of length 4″. In both these cases, the two right angles are positioned adjacent to each other, causing two corresponding sides of the triangles to form a straight line, producing a triangle as opposed to a quadrilateral. When adjoining the sides of lengh 5″, however, you once again obtain a quadrilateral, a kite.
- The 3″, 3″, 4″ isosceles triangles:
  - The three all-red quadrilaterals are parallelograms; the one obtained by lining up the 4″ sides is a rhombus with sides of length 3″.
  - There are two ways to form a mixed-color quadrilateral by adjoining sides of length 3″; one way produces a kite, while the other produces a parallelogram. The third mixed-color parallelogram is a rhombus with sides of length 3″, formed by lining up the two 4″ sides.

Problems 34–35 *(Student page 83)* Each all-red quadrilateral consists of two congruent triangles, one of which is obtained from the other by rotation and translation. Each mixed-color quadrilateral also consists of two congruent triangles, but now one is obtained from the other by a reflection. Notice that the mixed-color quadrilaterals are symmetric about at least one diagonal, while the red quadrilaterals are not.

MYSTERY FIGURES

Problem 1 *(Student page 84)* Look at the following picture:



Because *ABCD* is a square, $\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{DA}$.

By construction, $\overline{AH} \cong \overline{BE} \cong \overline{CF} \cong \overline{DG}$.

Moreover, since each angle in the original square is a right angle, it follows that $\angle HAB \cong \angle EBF \cong \angle FCG \cong \angle GDH$, since they are also all right angles. Therefore, we can apply the SAS postulate to conclude that

$$\triangle AHE \cong \triangle BEF \cong \triangle CFG \cong \triangle DGH.$$

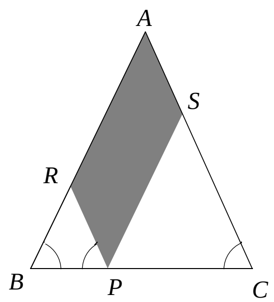
Now look at the labeled angles in the figure. By CPCTC, you know that $\angle 1 \cong \angle 3 \cong \angle 5 \cong \angle 7$ and $\angle 2 \cong \angle 4 \cong \angle 6 \cong \angle 8$. Putting the above two statements together, you see that

$$\angle GHE \cong \angle HEF \cong \angle EFG \cong \angle FGH,$$

which implies that *EFGH* is equiangular.

Finally, apply CPCTC to the four congruent triangles once again to see that $\overline{HE} \cong \overline{EF} \cong \overline{FG} \cong \overline{GH}$. This means that *EFGH* is also equilateral, so it must be a square.

Problem 2 *(Student page 84)* In isosceles triangle $\triangle ABC$, let P be any point on the base $\overline{BC}$, and construct segments $\overline{PR}$ and $\overline{PS}$ parallel to the congruent two sides of the triangle, as in the figure. The idea is to express the perimeter of $PRAS$ as a quantity which does not depend on P, R, or S. If you can do this, then the perimeter will not change as point P moves along the base $\overline{BC}$ (notice that as P moves, R and S also move).



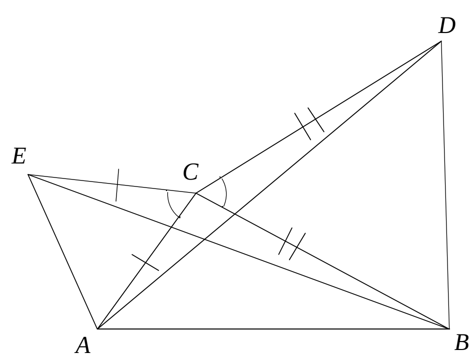
Because $\overline{SC}$ is parallel to $\overline{RP}$, it follows that $\angle SCP \cong \angle RPB$. But since $\triangle ABC$ is isosceles, it also follows that $\angle SCP \cong \angle RBP$, implying that $\angle RPB \cong \angle RBP$. This means that $\triangle RPB$ is isosceles, so $RB = RP$. Now write the perimeter of the parallelogram $PRAS$ as follows:

You proved that, if two angles of a triangle are congruent, the triangle is isosceles in Problem 10 of Investigation 2.9.

$$\begin{aligned}\text{Perimeter} &= 2(RA) + 2(RP)\\ &= 2(RA) + 2(RB)\\ &= 2(RA + RB)\\ &= 2(BA).\end{aligned}$$

Thus, the perimeter of $PRAS$ is equal to $2(BA)$, which will not change as the location of P changes.

Problem 3 *(Student page 84)* Consider the figure below.



By construction, $\overline{EC} \cong \overline{AC}$ and $\overline{CD} \cong \overline{CB}$. Also, $m\angle ECA = m\angle DCB = 60^\circ$, as they are both angles in equilateral triangles. You can use this equality to show that

$$m\angle ECA + m\angle ACB = m\angle DCB + m\angle ACB,$$

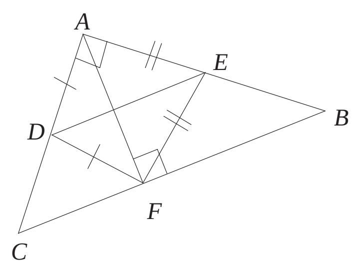
which implies that $\angle ECB \cong \angle DCA$. This is sufficient to conclude that $\triangle ECB \cong \triangle ACD$ by SAS; then CPCTC lets you conclude that $EB = AD$.

Problem 4 *(Student page 85)* This problem requires use of the following theorem:

THEOREM

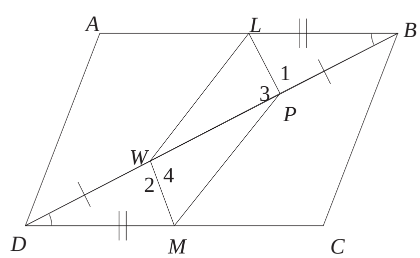
In a right triangle, the median to the hypotenuse is one half the length of the hypotenuse.

Let $\triangle ABC$ be a right triangle, as shown below, with the midpoints of the legs dentoted by D and E, and with F the point where the altitude from A intersects $\overline{BC}$.



Notice that $\overline{EF}$ is the median to the hypotenuse in right triangle $\triangle ABF$. The theorem then implies that $EF = AE$. Similarly, $\overline{FD}$ is the median to the hypotenuse in right triangle $\triangle FAC$, so $FD = AD$. Now we see that triangles $\triangle AED$ and $\triangle FED$ are congruent by SSS. It follows that $\angle EAD \cong \angle EFD$, which implies that $\angle EFD$ is a right angle, which is exactly what you want to show.

Problem 5 *(Student page 85)* In parallelogram $ABCD$, let W and P be points which trisect diagonal $\overline{DB}$, and let L and M be the midpoints of $\overline{AB}$ and $\overline{DC}$, respectively. The goal is to show that $LPMW$ is a parallelogram.



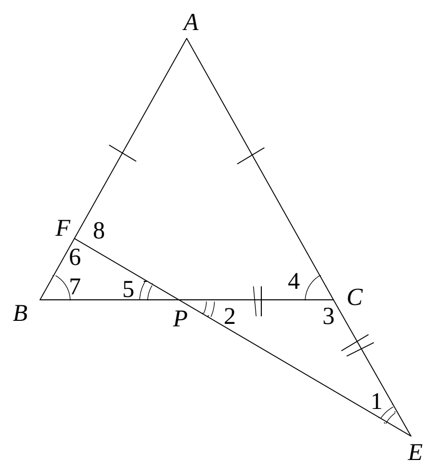
Notice that $\overline{DW} \cong \overline{BP}$, as each has length equal to one third the length of the diagonal.

MYSTERY FIGURES (continued)

Also, $\overline{MD} \cong \overline{LB}$, as each has half the length of one of two opposite sides of the large parallelogram $ABCD$. Moreover, $\angle LBP \cong \angle WDM$ because $\overline{AB} \parallel \overline{DC}$. This shows that $\triangle LBP \cong \triangle MDW$ by SAS. Then, by CPCTC, it follows that $\overline{LP} \cong \overline{WM}$, so two opposite sides of $LPMW$ are congruent. CPCTC also lets you conclude that $\angle 1 \cong \angle 2$.

Using this result, you see that $m\angle 1 + m\angle 3 = 180^\circ = m\angle 2 + m\angle 4$, implying that $\angle 3 \cong \angle 4$. This means that $\overline{LP} \parallel \overline{WM}$. Since two sides are parallel and congruent, it follows that $LPMW$ is a parallelogram.

Problem 6 *(Student page 85)* This problem leads to the following picture. (We are trying to show that $m\angle 8 = 3\,m\angle 1$.)



Since $\triangle ABC$ is isosceles, $\angle 4 \cong \angle 7$. Also, $\angle 5 \cong \angle 2$ since they are vertical angles, and since $\overline{PC} \cong \overline{CE}$ by construction, it follows that $\triangle CPE$ is isosceles, so $\angle 2 \cong \angle 1$.

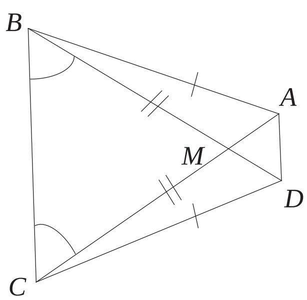
Because the sum of the measures of angles in a triangle is 180°, you can look at $\triangle PCE$ and conclude that $m\angle 1 + m\angle 2 + m\angle 3 = 180^\circ$. But $m\angle 1 = m\angle 2$, so substitute above to see that $2m\angle 1 + m\angle 3 = 180^\circ$.

Furthermore, $m\angle 3 + m\angle 4 = 180^\circ$ as these angles form a straight angle. Therefore, $180^\circ = m\angle 3 + m\angle 4 = 2m\angle 1 + m\angle 3$, so $m\angle 4 = 2m\angle 1$.

In $\triangle FBP$, $m\angle 5 + m\angle 6 + m\angle 7 = 180^\circ$. But $m\angle 5 = m\angle 2 = m\angle 1$ and $m\angle 7 = m\angle 4 = 2m\angle 1$, so $m\angle 1 + m\angle 6 + 2m\angle 1 = 180^\circ$ and $3m\angle 1 = 180^\circ - m\angle 6$.

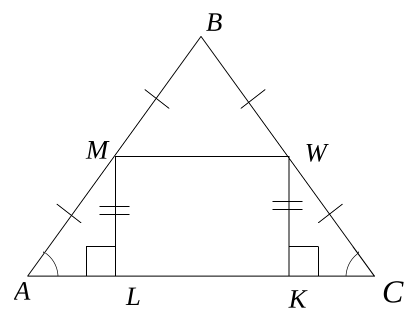
MYSTERY FIGURES (continued)

Since $180^\circ - m\angle 6 = m\angle 8$, it follows that $3m\angle 1 = m\angle 8$, which is exactly what we needed to show.



Problem 7 *(Student page 85)* Suppose that in the quadrilateral shown here, sides $\overline{AB}$ and $\overline{DC}$ are congruent, as well as diagonals $\overline{AC}$ and $\overline{DB}$. This is enough to conclude that $\triangle DCB \cong \triangle ABC$ by SSS, since they share side $\overline{CB}$. Then you can conclude that $\angle ACB \cong \angle DBC$ by CPCTC, implying that $\triangle MBC$ is isosceles, where M is the intersection point of the two diagonals.

Problem 8 *(Student page 85)* This problem results in the picture below.



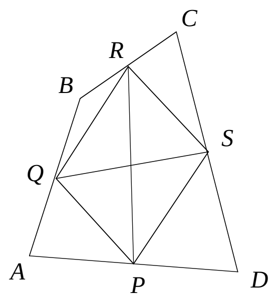
Besides the given information, let M be the midpoint of $\overline{AB}$ and construct an altitude from M to $\overline{AC}$, calling the point of intersection L. Because $\overline{AB} \cong \overline{BC}$ and M and W are midpoints, it follows that $\overline{AM} \cong \overline{CW}$. Also, you know that $\angle MAL \cong \angle WCK$ since $\triangle ABC$ is isosceles, and $\angle MLA \cong \angle WKC$ as they are both right angles. This means that $\triangle MLA \cong \triangle WKC$ by AAS. Thus, $\overline{LA} \cong \overline{KC}$.

Now recall the Midline Theorem, which states that the segment connecting the midpoints of two sides of a triangle is parallel to the third side and half its length. Because $\overline{MW} \parallel \overline{AC}$, it follows that $LMWK$ is a rectangle, implying that $MW = LK$. This means, however, that $LK = \frac{1}{2}AC$.

Therefore, it follows that $AC = AL + LK + KC$, so $AC = KC + \frac{1}{2}AC + KC$, and therefore $\frac{1}{2}AC = 2KC$.

This implies that $\frac{1}{4}AC = KC$.

Problem 9 *(Student page 85)* We first show that, if you connect in order the midpoints of the sides of any quadrilateral in order, you get a parallelogram. Then the line segments connecting opposite midpoints will be the diagonals of this parallelogram, and so will bisect each other (see Problem 14 of Investigation 2.9).



Let P, Q, R, and S be the midpoints of the sides of quadrilateral $ABCD$, and form quadrilateral $PQRS$. Now, $\overline{QR}$ is a midline of $\triangle ABC$, so the Midline Theorem says that it is parallel to $\overline{AC}$. Furthermore, $\overline{PS}$ is a midline of $\triangle ADC$, so it is also parallel to $\overline{AC}$, implying that $\overline{QR}$ and $\overline{PS}$ are parallel to each other. In exactly the same way, show that $\overline{QP}$ and $\overline{RS}$ are parallel, so $PQRS$ is a parallelogram.

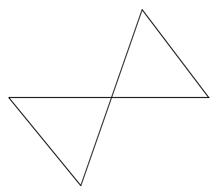
BEYOND TRIANGLES

For Discussion *(Student page 86)* It is easy to find real-world examples of squares and rectangles, and for kites, as the name implies. Many drawings and paintings contain parallelograms; for example a wall or a door might be drawn as a parallelogram to add perspective to the picture. Many drawings by the artist Escher are filled with geometric objects.

Problems 1–11 *(Student pages 86–87)* To help you with these problems, here's a list of definitions of various quadrilaterals, along with properties that you might notice to be true. You may know many of these properties already, and may have been using them in your proofs, but this is a good time to make an organized list.

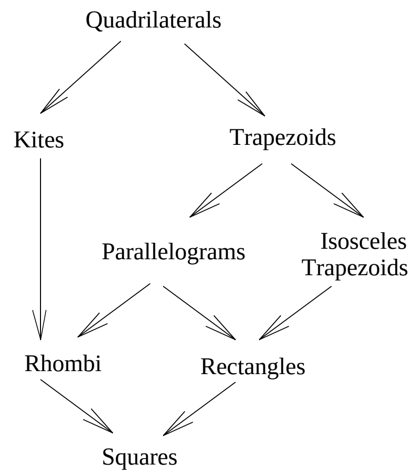
Problems 1 and 2 have many solutions. Problems 3, 5, 6, 7, 8, 9, and 10 have only one solution. Problem 4 has two solutions.

- A parallelogram is a quadrilateral with both pairs of opposite sides parallel.
  - Opposite sides are congruent.
  - Opposite angles are congruent.
  - Consecutive angles are supplementary.
  - The diagonals bisect each other.
- A rectangle is a quadrilateral with all angles congruent.
  - A rectangle is a parallelogram, so has all the properties listed above.
  - All angles are 90°.
  - The diagonals are congruent.
- A rhombus is a quadrilateral with all sides congruent.
  - It is a parallelogram, with all the corresponding properties holding.
  - The diagonals are perpendicular to each other.
  - The diagonals bisect the angles.
- A kite is a quadrilateral with two pairs of adjacent sides congruent.
  - The diagonal which cuts between both congruent pairs of sides divides the kite into two congruent triangles.
  - The two angles between the noncongruent sides are congruent.
- A trapezoid is a quadrilateral with two sides parallel.
- An isosceles trapezoid is a trapezoid with congruent base angles.

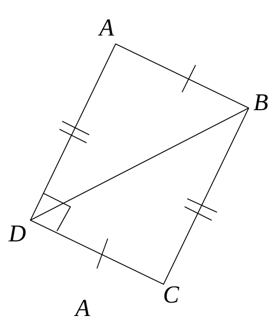


Is this a quadrilateral?

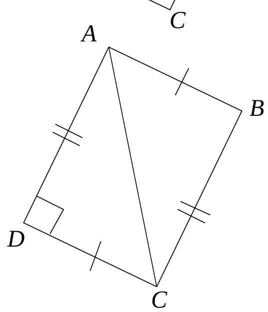
Problem 12 *(Student page 87)* The diagram below shows one way to classify quadrilaterals.



Problem 13 *(Student page 88)* In this problem, you have a quadrilateral containing one right angle and having opposite sides of equal length; you want to show that the remaining three angles are also right angles.



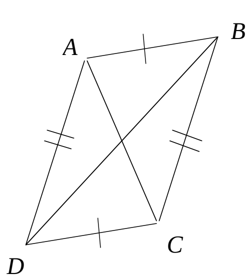
Suppose you have quadrilateral $ABCD$ with $m\angle D = 90^\circ$, $\overline{AB} \cong \overline{DC}$, and $\overline{DA} \cong \overline{CB}$. Draw diagonal $\overline{DB}$. You can use SSS to see that $\triangle ABD \cong \triangle CDB$, which implies that $\angle A \cong \angle C$.



Now draw diagonal $\overline{AC}$. The SSS postulate shows that $\triangle ADC \cong \triangle CBA$, which implies that $\angle D \cong \angle B$, so $\angle B$ is a right angle.

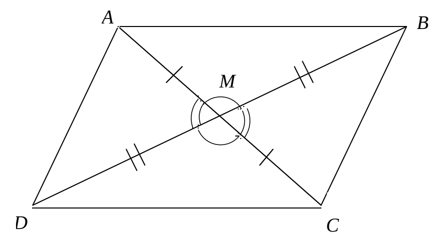
Since the sum of the measures of the angles in a quadrilateral is 360°, $m\angle A + m\angle B + m\angle C + m\angle D = 360^\circ$. But using the angle congruence above, you see that this means $2m\angle A + m\angle B + m\angle D = 360^\circ$. So $2m\angle A + 2(90^\circ) = 360^\circ$.

Therefore, $\angle A$ must be a right angle, and so must $\angle C$, as they are congruent. So all four angles are right angles, and the quadrilateral is a rectangle.



Problem 14 *(Student page 88)* The argument used in the previous problem can be modified to show that, if opposite sides of a quadrilateral are congruent, then they must also be parallel. Suppose $ABCD$ is a quadrilateral with opposite sides parallel. As above, you can show that $\triangle ADC \cong \triangle CBA$, which means that $\angle CAD \cong \angle ACB$. This shows that sides $\overline{AD}$ and $\overline{BC}$ are parallel. Similarly, you can show that $\triangle ABD \cong \triangle CDB$, and conclude that sides $\overline{AB}$ and $\overline{CD}$ are parallel.

Problem 15 *(Student page 88)* If a quadrilateral's diagonals bisect each other, the quadrilateral must be a parallelogram. Suppose quadrilateral $ABCD$ is given, with M the point of intersection of the diagonals. Thus you know that $\overline{AM} \cong \overline{MC}$ and $\overline{DM} \cong \overline{MB}$.

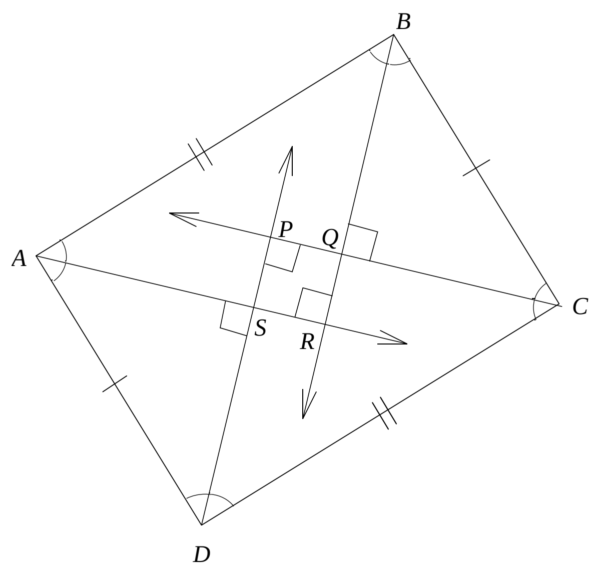


Because $\angle DMC$ and $\angle AMB$ are congruent vertical angles, the SAS postulate gives that $\triangle DMC \cong \triangle BMA$; because $\angle AMD$ and $\angle BMC$ are congruent vertical angles, it follows that $\triangle AMD \cong \triangle CMB$, again by SAS. The first triangle congruence gives that $\overline{DC} \cong \overline{BA}$, while the second congruence gives that $\overline{AD} \cong \overline{BC}$. Therefore, the opposite sides are congruent; we saw in the previous problem that this implies that the quadrilateral is a parallelogram.

Problem 16 *(Student page 88)* See the notes for Problem 7 of Investigation 2.8 for a proof that a quadrilateral is a parallelogram if two opposite sides are congruent and parallel.

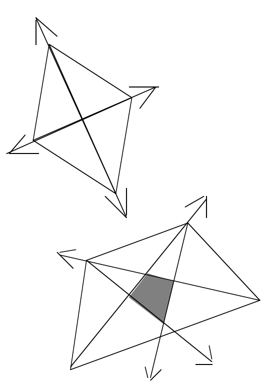
Problem 17 *(Student page 88)* A parallelogram with two consecutive sides congruent will also automatically be a rhombus. This is because a parallelogram has opposite sides congruent, but if consecutive sides are also congruent, then the parallelogram is equilateral, and hence a rhombus.

Problems 18–19 *(Student page 89)* In a rectangle, the angle bisectors intersect to form a square. Suppose you start with rectangle *ABCD*, and let *PQRS* be the quadrilateral formed by the angle bisectors.



Since *ABCD* is a rectangle, you know that $\overline{AB} \cong \overline{DC}$ and $\overline{AD} \cong \overline{BC}$. Moreover, you know that each angle bisector produces two 45° angles. This is enough to conclude that $\triangle ASD \cong \triangle BQC$ and $\triangle ABR \cong \triangle DCP$, both by ASA. Further, each of these four triangles is isosceles, since the 45° base angles are congruent. This means that $\overline{AS} \cong \overline{SD} \cong \overline{BQ} \cong \overline{QC}$ and $\overline{AR} \cong \overline{BR} \cong \overline{DP} \cong \overline{CP}$.

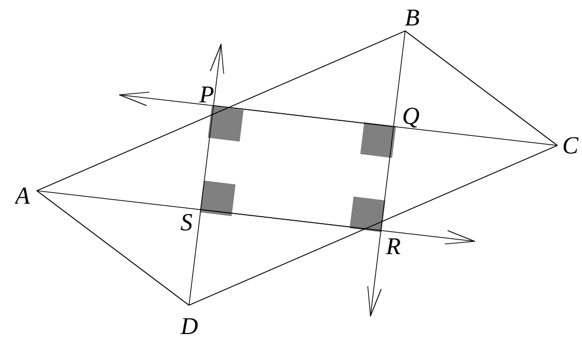
For squares and rhombi, the angle bisectors are concurrent, and for isosceles trapezoids, the angle bisectors form a kite.



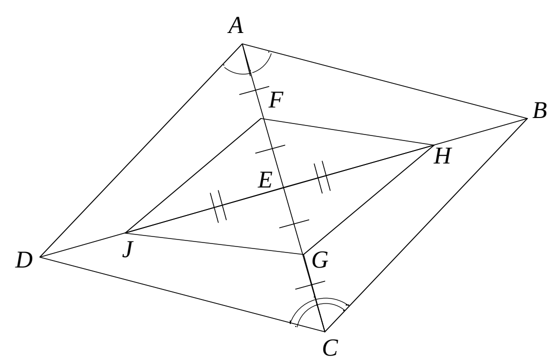
You now know that $\angle ASD$ and $\angle BQC$ are right angles; since vertical angles are congruent, $\angle PSR$ and $\angle PQR$ are also right angles. Notice also that $\angle SRQ$ and $\angle SPQ$ are also right angles; this can be seen by looking at triangles $\triangle ABR$ and $\triangle DPC$. Therefore, all angles of *PQRS* are right angles, so it is a rectangle.

Now, notice that $AR = AS + SR$ and $BR = BQ + QR$. Since $AR = BR$ and $AS = BQ$ from the congruences above, it follows that $SR = QR$, so $\overline{SR} \cong \overline{QR}$. But, if consecutive sides of a rectangle are congruent, the rectangle must be a square, so *PQRS* is a square.

Suppose the quadrilateral *ABCD* that you start with is a parallelogram. In this case, the angle bisectors intersect to form a rectangle. This can be shown in basically the same fashion as above.



Problem 20 *(Student page 89)* First of all, notice that the diagonals of *FHGJ* bisect each other. This implies that *FHGJ* is a parallelogram (see the notes for Problem 8 of Investigation 2.8). The SSS postulate then implies that $\triangle FEJ \cong \triangle GEH$ and $\triangle JEG \cong \triangle HEF$.



Now look at *ABCD*. Since $\angle DAE \cong \angle BAE$ and $\angle DCE \cong \angle BCE$, triangles $\triangle ADC$ and $\triangle ABC$ are congruent by ASA. This means that *ABCD* is a kite, with $AD = AB$ and $DC = BC$. But then, SAS implies that $\triangle ADE \cong \triangle ABE$ and $\triangle CDE \cong \triangle CBE$.

This last triangle congruence implies that $\angle CED \cong \angle CEB$. But $\angle AEB$ and $\angle CED$ are vertical angles, so $\angle AEB \cong \angle CEB$.

BEYOND TRIANGLES (continued)

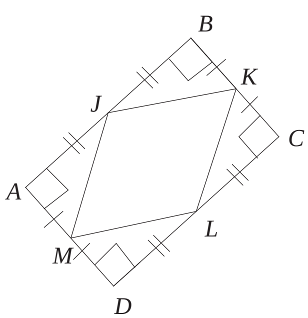
This, along with the fact that $\overline{AE} \cong \overline{CE}$ and $\overline{EB} \cong \overline{EB}$, shows that $\triangle ABE \cong \triangle CBE$, by SAS.

Thus, all four triangles of *ABCD* are congruent:

$$\triangle ADE \cong \triangle ABE \cong \triangle CDE \cong \triangle CBE.$$

This implies that *ABCD* is actually a rhombus.

Finally, you now also know that all four angles at E are right angles; this is because *ABCD* is a rhombus, so its diagonals intersect perpendicularly. Therefore, *FHGJ* is a parallelogram whose diagonals intersect at right angles, which means it too is a rhombus.



Problem 21 *(Student page 89)* A rhombus is formed by connecting the midpoints of a rectangle's sides. If *ABCD* is a rectangle with J, K, L, and M the midpoints of the sides, then the SAS postulate shows that

$$\triangle AJM \cong \triangle BJK \cong \triangle CLK \cong \triangle DLM.$$

Then the following corresponding sides are all congruent: $\overline{JM} \cong \overline{JK} \cong \overline{LK} \cong \overline{LM}$. Thus, *JKLM* is a rhombus.

Problem 22 *(Student page 89)*

a. Quadrilateral *CJEH* is a rectangle. Because a rhombus is a parallelogram, you know that $\overline{CF} \parallel \overline{DE}$. This means that $\angle JCH \cong \angle CHD$, as they are alternate interior angles formed by the transversal $\overline{CH}$. This tells you that $\angle JCH$ is a right angle, and because the sum of the measures of the angles in any quadrilateral is 360°, it follows that $\angle CJE$ is also a right angle. Therefore, *CJEH* is a rectangle, because it is a quadrilateral with four right angles.

b. You can prove that $\triangle GLE \cong \triangle GKC$ using the AAS postulate. First of all, $\overline{CG} \cong \overline{EG}$, since the diagonals of a parallelogram bisect each other. Then, since *CJEH* is a rectangle, sides $\overline{JE}$ and $\overline{CH}$ are parallel, so we can find congruent alternate interior angles. In particular, it follows that $\angle GEL \cong \angle GCK$ and $\angle ELG \cong \angle GKC$. This is enough to apply AAS and conclude that the triangles are congruent.

c. Here is a list of several other triangle congruences that can be proven from the given figure.

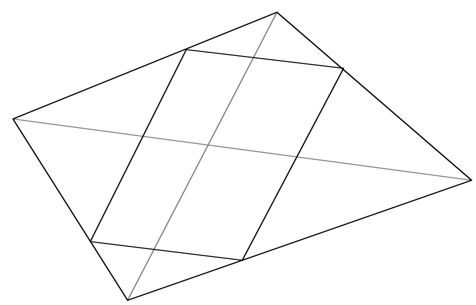
$\triangle DCF \cong \triangle DEF$	$\triangle CFE \cong \triangle CDE$
$\triangle FCG \cong \triangle FEG$	$\triangle DCG \cong \triangle DEG$
$\triangle CHE \cong \triangle EJC$	$\triangle CKD \cong \triangle ELF$
$\triangle JFL \cong \triangle HDK$	$\triangle DHC \cong \triangle FJE$

Problem 23 *(Student page 90)*

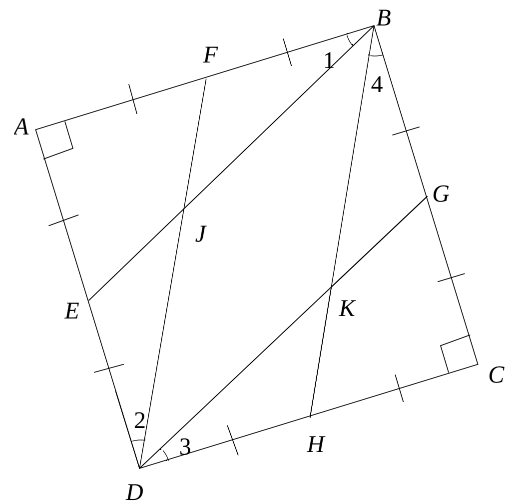
a. You can use the SAS postulate to show that $\triangle QXW \cong \triangle NTS$ and $\triangle MSX \cong \triangle PWT$. Then by CPCTC you see that $\overline{XW} \cong \overline{TS}$ and $\overline{SX} \cong \overline{WT}$.

b. The above shows that *STWX* is a parallelogram, since opposite sides are congruent. Therefore, any of the parallelogram properties will hold.

c. If *MNPQ* were not a parallelogram, *STWX* would still be one, but we could not use SAS to prove it. We would have to use a generalization of the Midline Theorem.



Problem 24 *(Student page 90)* The picture includes all of the following quadrilaterals: *AFJE*, *ABHD*, *DJBK*, *DJBH*, *CHKG*, *CDFB*, *FBGD*, *EDHB*, *EDKB*, *EDGB*, *DHBF*, *DKBF* and *JBGDG*.



Notice that by construction, the segments $\overline{AF}$, $\overline{FB}$, $\overline{BG}$, $\overline{GC}$, $\overline{CH}$, $\overline{HD}$, $\overline{DE}$, and $\overline{EA}$ are all congruent. Notice also that by the SAS postulate, the following triangles are all congruent: $\triangle ABE$, $\triangle CBH$, $\triangle ADF$, and $\triangle CDG$. These congruent triangles can be used to conclude that angles $\angle 1$, $\angle 2$, $\angle 3$, and $\angle 4$ are all congruent. Further, notice the pairs of vertical angles, $\angle EJD \cong \angle FJB$ and $\angle DKH \cong \angle BKG$.

All of the above congruence information can be used to give the following classification of the various quadrilaterals.

- *EBGD* is a parallelogram, since opposite sides $\overline{BG}$ and $\overline{ED}$ are congruent and parallel.
- *EDKB* is a trapezoid, since $\overline{EB} \parallel \overline{DK}$.
- *JBGD* is a trapezoid, since $\overline{JB} \parallel \overline{DG}$.
- *DHBF* is a parallelogram, since opposite sides $\overline{DH}$ and $\overline{FB}$ are congruent and parallel.
- *DJBH* is a trapezoid, since $\overline{DJ} \parallel \overline{HB}$.
- *DKBF* is a trapezoid, since $\overline{DF} \parallel \overline{KB}$.
- *AFJE* is a kite. The AAS postulate can be used to show $\triangle EDJ \cong \triangle FBJ$, giving that $\overline{EJ} \cong \overline{FJ}$. This, combined with the fact that $\overline{AE} \cong \overline{AF}$, gives the kite.
- Similarly, *CHKG* is a kite. Use the AAS postulate to show that $\triangle DHK \cong$

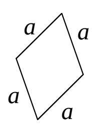
BEYOND TRIANGLES (continued)

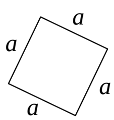
$\triangle BGK$, giving that $\overline{HK} \cong \overline{GK}$. This, plus the fact that $\overline{HC} \cong \overline{GC}$, shows that you have a kite.

- *FBGD* is also a kite. The fact that $\triangle AFD \cong \triangle CHB$ implies that $\overline{FD} \cong \overline{HB}$. Combine this with the congruence $\overline{FB} \cong \overline{BG}$.
- It follows in the same manner that *EDHB* is also a kite. Use the congruence $\triangle ABE \cong \triangle CBH$ to give that $\overline{BE} \cong \overline{BH}$. Combine this with the congruence $\overline{DE} \cong \overline{DH}$ to get the kite.
- *DJBK* is a rhombus. It is clearly a parallelogram; you can conclude that its opposite sides are parallel using the previously-given parallelograms. This means that opposite sides are congruent. But, recall that $\triangle EDJ \cong \triangle FBJ$. This means that $\overline{DJ} \cong \overline{BJ}$, implying that all four sides of *DJBK* are congruent.

Problem 25 *(Student page 92)* There are three orders in which to arrange the sidelengths. Suppose that you always start at the side of length 5, listing the length of each side. The different orders you can get are:

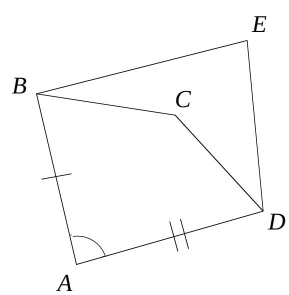
- 5, 6, 7, 8
- 5, 6, 8, 7
- 5, 7, 6, 8.

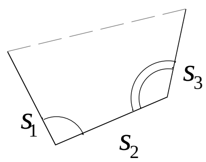




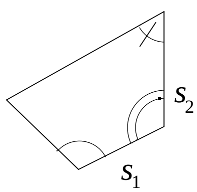
Problem 26 *(Student page 92)* Suppose that you have a rhombus which is not a square, and which has all sides of length a, for some number a. This quadrilateral will not be congruent to the square with sidelength a, so there is no SSSS test for congruent quadrilaterals.

Problem 27 *(Student page 92)* SAS is not a congruence test for quadrilaterals; it is possible to find different quadrilaterals sharing two sides and an included angle of the same measure, but which are not congruent. For example, in the figure below, quadrilaterals *ABCD* and *ABED* share sides $\overline{AB}$ and $\overline{AD}$ along with the included $\angle A$, but the two figures are clearly not congruent.

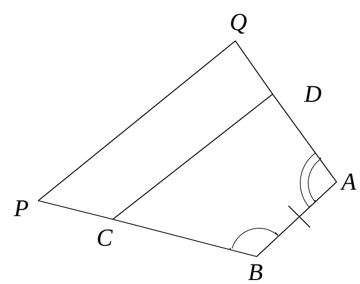




One possible congruence test is SASAS—three consecutive sides along with the two included angles. By trying some examples, you can see that, if you fix these quantities (the two marked angles, plus the three sides denoted by s_1, s_2, and s_3), there is no choice about where to put the fourth side (it must be the dashed segment). Therefore, the quadrilateral is completely determined.



Problem 28 *(Student page 92)* There is also no ASA test for quadrilaterals. In the figure below, quadrilaterals $ABCD$ and $ABPQ$ share $\angle A$ and $\angle B$, plus side $\overline{AB}$, but are not congruent.



You can devise an ASASA test, however. Once again, if these five quantities are fixed, the quadrilateral is completely determined.

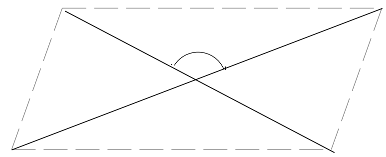
Problem 29 *(Student page 92)* A rectangle is uniquely determined by the lengths of its base and height. To check this, notice that all angles in a rectangle must be right angles, and opposite sides are congruent. So once you know the lengths of two sides, all other pieces of the rectangle are known. Thus, BH is a test for rectangle congruence.

Problem 30 *(Student page 92)*

a. Two parallelograms are congruent if they have congruent diagonals and one of the angles of intersection of the diagonals is the same size in both parallelograms. By knowing the lengths of both diagonals and one angle between them, you automatically know the position of the diagonals. Then you can just connect the endpoints of these diagonals to form the parallelogram.

Note that you *must* know that the figures are parallelograms. Congruent diagonals with the angle of intersection is not enough to show congruence for any quadrilateral, but in parallelograms you also know that the diagonals bisect each other.

Since this information completely determines a parallelogram, it is also a congruence test for two parallelograms.



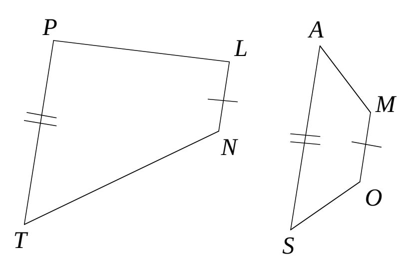
b. Two rhombi are congruent if they have one pair of congruent sides and one pair of corresponding angles congruent. By definition, a rhombus is equilateral, so knowing that one pair of sides is congruent guarantees that all four sides are congruent. Also, if the angles in one pair agree in measure, then the opposite angles agree (all opposite angles in a rhombus are congruent), and the remaining two angles must also agree (as adjacent angles are supplementary). In other words, the measure of one angle completely determines the measures of the remaining three angles.

c. Two trapezoids are congruent if they have the same base angles and height, as well as the same base length.

Problem 31 *(Student page 93)* In all three of these questions, the given information is not sufficient to determine that the quadrilaterals are congruent.

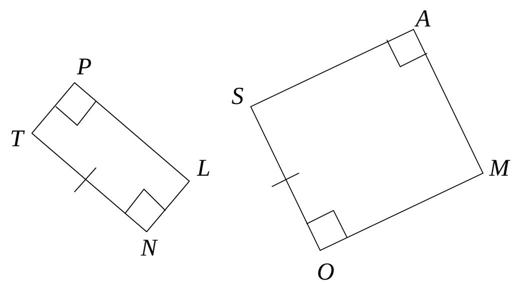
Remember that to show the information is not sufficient, it is enough to find one example in which the information holds, but the figures are not congruent.

a. Pictured below are two quadrilaterals satisfying the properties that $PT = AS$, $LN = MO$, and segments $\overline{PT}$ and $\overline{LN}$ are parallel, as well as segments $\overline{AS}$ and $\overline{MO}$. The two quadrilaterals are not congruent, however, so the given information does not produce a sufficient quadrilateral test.



b. The counterexample shown here is two rectangles with the same width but different heights. $PLNT$ and $AMOS$ below are both rectangles; therefore $\angle P$ and $\angle A$ have the same measure, as do $\angle N$ and $\angle O$. Moreover, the two

rectangles have the same width, so $TN = SO$. As you can see, though, the rectangles are not congruent.



c. Suppose that *PLNT* is a nonsquare rhombus with sides of length 1”, while *AMOS* is a square with sides of length 1”. Then these quadrilaterals will satisfy the given properties, but will not be congruent.

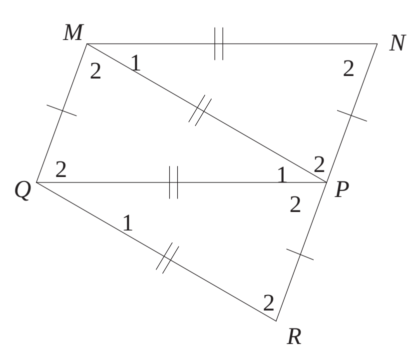
Problem 32 *(Student page 94)*

a. Yes, parallelograms *MNPQ* and *MPRQ* are congruent. First, since opposite sides of a parallelogram are congruent, you know that $\overline{MN} \cong \overline{QP}$, $\overline{MP} \cong \overline{QR}$, and $\overline{MQ} \cong \overline{NP} \cong \overline{PR}$.

Since *MPRQ* is a parallelogram, it follows that $\overline{MP} \cong \overline{RQ}$, which means that $\angle MPQ \cong \angle PQR$. You also know that $\overline{MQ} \cong \overline{PR}$. Since *MNPQ* is a parallelogram, you know that $\angle QMP \cong \angle QPR$.

b. The above information shows, by AAS, that $\triangle PQM \cong \triangle QRP$. This congruence implies that $\overline{QP} \cong \overline{QR}$, so combining with the above information, we have four congruent segments:

$$\overline{MN} \cong \overline{QP} \cong \overline{QR} \cong \overline{MP}.$$



This is enough information to conclude that $\triangle MNP$, $\triangle PQM$, and $\triangle QRP$ are all congruent to each other, and, moreover, they are all isosceles. This means that all angles numbered 1 in the diagram above are congruent to each other, as are all angles numbered 2. Now, you can see that all corresponding sides and angles of parallelograms *MNPQ* and *MPRQ* are congruent, so the parallelograms are, in fact, congruent.

c. Because the sum of the measures of the angles in a triangle is 180°, it follows that $2\,m\angle 2 + m\angle 1 = 180^\circ$. Look at $\angle NPR$, which can be broken up as $\angle NPR = \angle NPM + \angle MPQ + \angle QPR$, so

$$m\angle NPR = m\angle 2 + m\angle 1 + m\angle 2.$$

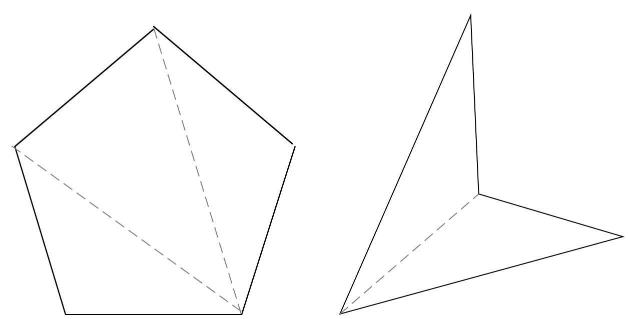
Therefore, $m\angle NPR = 180^\circ$, which means N, P, and R are collinear.

d. If $\angle N$ measures 70°, then all angles labeled 2 also measure 70°, and all angles labeled 1 measure 40°.

e. If $MN = 4.0$ cm and $PR = 2.5$ cm, then the perimeter of *MNPRQ* is given by $2(4) + 3(2.5) = 15.5$ cm.

Problem 33 *(Student page 94)*

The key thing to remember here is that, from your previous work, you know you can completely divide any n-gon up into $(n-2)$ triangles by drawing each of the diagonals originating at one vertex. Because of SSS congruence in triangles, if you know all the sides and all of the diagonals from a single vertex, you have completely determined the polygon.



Here, the five sides of the pentagon plus the two diagonals from one vertex completely determine the pentagon. Similarly, the four sides of a quadrilateral plus one diagonal completely determine that quadrilateral.

If you know all of the lengths of the sides of two n-gons, and if you know $(n - 3)$ diagonals all sharing a corresponding common vertex, you can determine congruence between these two n-gons.

This doesn't quite work for concave polygons. It might matter where you start, and there may not be a single vertex whose diagonals work. Can you fix the solution to account for this? Does the number of diagonals change?

Congruence in Three Dimensions

Problem 1 *(Student page 95)* Two three-dimensional figures are congruent if, as always, they have the same size and shape. One way to think of this is that two solids have the same size if they have the same volume. So two three-dimensional figures with the same dimensions and volume are congruent.

Problem 2 *(Student page 95)* Two spheres with the same volume are congruent. In fact, any two spheres with the same radius are congruent. Congruent spheres, being the same size, have the same surface area. Note that the formulas for volume and surface area of a sphere are given in terms of the radius; thus the radius is the only measurement you need to know to completely determine the sphere. (The volume of a sphere is $\frac{4}{3}\pi r^3$ and the surface area is $4\pi r^2$.)

Problem 3 *(Student page 95)* The two rectangular boxes have the same height and width, but they could have different lengths. Suppose you have two boxes; their dimensions (length, width, and height) are $4'' \times 5'' \times 12''$ and $1'' \times 5'' \times 12''$. Both boxes are 12 inches tall and 5 inches wide, but they are not congruent. Notice that the first one's volume is 240 cubic inches, while the second has a volume of 60 cubic inches; they are definitely not the same size!

Problem 4 *(Student page 95)* The two boxes have the same height and volume, so they must have the same base area. But they need not be congruent. Suppose the two boxes are each 10 cm tall and each contain 100 cm^3 of cereal. The base area must be 10 cm^2, but the dimensions of the base could be 5×2 or 1×10 or . . . .

Problem 5 *(Student page 95)* If two noncongruent cylinders have the same height, they must have different radii, as a cylinder is completely determined by its height and its radius. Therefore, the surface areas, the circumferences of the bases, and the volumes of the two cylinders will also differ, as these three quantities depend on the radius. (The surface area of a cylinder is $2\pi rh$, the circumference of the base is $2\pi r$, and the volume is $\pi r^2 h$.)

Problem 6 *(Student page 95)* All cubes are not congruent, but all cubes with the same sidelength are congruent. Therefore, the minimum number of measurements needed is just one, the length of a side.

Problem 7 *(Student page 95)* A tetrahedron with all faces equilateral will have six edges, as will any tetrahedron. Because each face is an equilateral triangle, however, there will be only one edge *length*; all edges are congruent. There are two different angle sizes which can be measured. The angles lying on each face will all measure 60°, but you also need to consider the interior angles of the tetrahedron. These are the angles at which the faces meet, or in other words, the angles at which the planes

How do you measure the angle between two planes?

containing the faces meet; these angles will all be congruent, but not necessarily measure 60°. Thus, there are two different angle sizes.

Problem 8 *(Student page 95)* Consider each face separately, measuring each edge and all diagonals. This will uniquely determine the face, so you can match up corresponding faces. But we know each face determines the box structure, so this is sufficient.

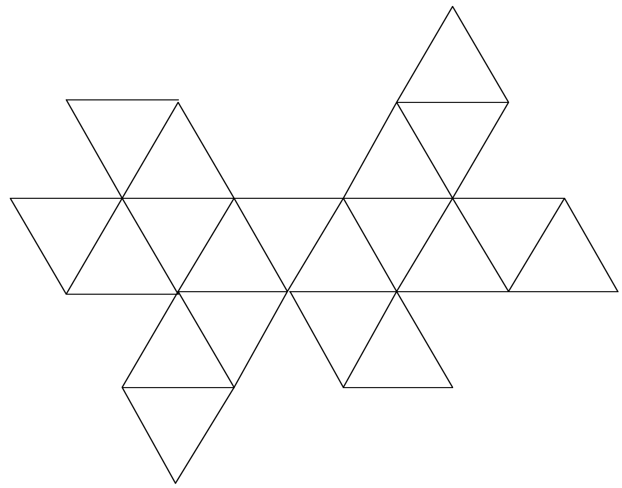
Problem 9 *(Student page 95)* The nets in Set A can be folded to form congruent rectangular boxes; the boxes are long and skinny, with a cover on each end. All three nets in Set B form pyramids consisting of a rectangular base with four triangular sides. The nets in Set C both fold to congruent antiprisms with square bases. Finally, each net in Set D can be folded into a cube.

Figures with congruent bases and rectangular sides are *prisms*. Figures with congruent bases and triangular sides are *antiprisms*.

Problem 10 *(Student page 97)* The two nets below will fold to make congruent triangular prisims.

CONGRUENCE IN THREE DIMENSIONS (continued)

Problem 11 *(Student page 97)* The net below can be folded into two noncongruent solids. First, fold along all the edges to create a convex solid. Then try to fold a second copy of the net, but this time add a concavity (indentation).



For Discussion *(Student page 97)* Different types of solids have different parts. Some of these parts include faces, edges, diagonals, heights, widths, radii, and bases.

Problem 12 *(Student page 97)*

a. Two cubes are congruent if they have the same height; you can omit "surface area and." Also, two cubes are congruent if they have the same surface area; you can omit "and height."

In fact, any combination of three from the set length, width, height, and volume will work.

b. Two rectangular solids are congruent if they have the same length, width, and height; you don't need the volume.

c. This statement contains just the right amount of information, nothing extra. It is possible to have two noncongruent square pyramids with the same height and base length, so you need the fact that in this case the two pyramids have the same slant edge length.

d. Two cylinders are congruent if they have the same height and circumference (or same height and surface area, or same surface area and circumference).

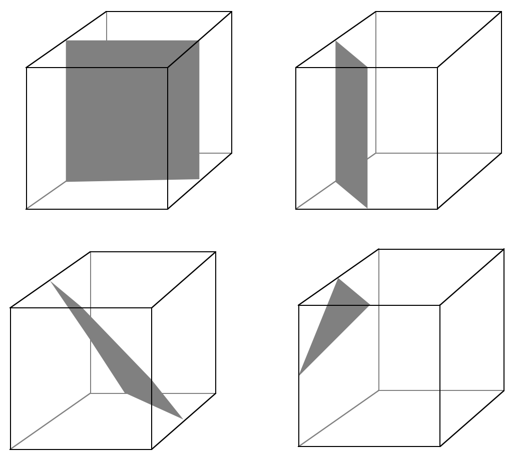
Remember that a regular tetrahedron is one in which all four faces are congruent equilateral triangles. This is a pretty strict condition, which is why you don't need a lot of information to decide if two of them are congruent.

e. Two regular tetrahedra are congruent if they have the same height or if they have the same volume. You don't need to know both conditions.

Problem 13 *(Student page 98)*

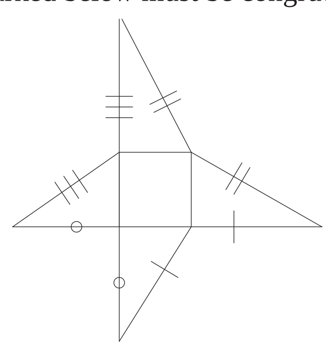
a. The plane will cut the cube into two congruent solids. You can check that all corresponding angles and edges will have the same measurements.

b. Here are four different ways in which a plane may intersect a cube only at the midpoints of its edges. In the two pictures on the left, the plane split the cube into two congruent solids.



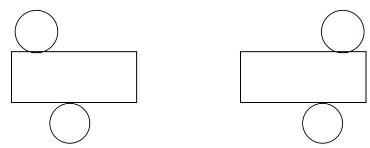
c. By definition, any two congruent solids will have equal volumes and surface areas.

Problem 14 *(Student page 99)* In order for this net to fold into a rectangular pyramid, the segments marked below must be congruent.



Problem 15 *(Student page 99)* In the development for a cylinder, the two circles must have the same radius (hence be congruent), and the circumference of the circles must be equal to the width of the rectangle.

Problem 16 *(Student page 99)* The two developments shown below give the same cylinder. The two rectangles are congruent, and all four circles are congruent. Since a cylinder is determined by its radius and height, the two resulting cylinders will be congruent.



The sphere, soup bowl, football, and bell of a trumpet cannot be made because of the constant curvature of the surface.

Problem 17 *(Student page 99)* A cone, body of a guitar, and playground slide can all be built from plane figures. Pictures for the development of each of these are shown below. For the guitar, the length of the rectangle should match the perimeter of the curved body, but, to save space, we have clipped the picture a bit.